Arithmetic of L-functions.

Ex. $E: y^2 = x^3 - 25x$ $\qquad E^{(n)}: y^2 = x^3 - n^2 x$.

$r_{alg}(E^{(n)})$	n
0	1
1	5
2	34
3	1254
4	29274.
5	48272239
6	6611719866
7	?

BSD conj. Analogues of Riemann Zeta fctns:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_n \frac{1}{n^s}.$$

define $L(E,s)$ associated with $E/\mathbb{Q}$.

$$L(E,s) = \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p L_p(E,s).$$

where
$$L_p(E,s) = \begin{cases} (1 - a_p \cdot p^{-s} + p^{1-2s}) & p \text{ good.} \\ (1 - a_p \cdot p^{-s}) & p \text{ bad.} \end{cases}$$

$$a_p = \begin{cases} p+1 - E(\mathbb{F}_p) & p \text{ good} \\ \pm 1 & p \text{ multiplicative} \\ 0 & p \text{ additive.} \end{cases}$$

$L(E,s)$ converges when $\mathrm{Re}(s) > \frac{3}{2}$.

Modularity Thm (Wiles, Taylor, BCDT) $\exists$ a cuspidal eigenform $f_E \in S_2(N_E)$ associated w/ E s.t.

$$f_E(z) = \sum_{n \geq 1} a_n q^n \text{ as before.}$$

Ex. $y^2 = x^3 - x$ $(N_E = 32)$.

p	2	3	5	7	11	13	17	19	23	29	31
a_p	0	0	-2	0	0	6	2	0	0	-10	0

(notice $a_p \neq 0 \iff p \equiv 1 \pmod 4$).

$$f_E = q \prod_{n \geq 1} (1 - q^{4n})^2 (1 - q^{8n})^2.$$
$$= q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} - q^{25} - 10q^{29} + \cdots$$

Cor. $L(E,s) = L(f_E, s)$. In particular, $L(E,s)$ has analytic continuation to $\mathbb{C}$ and satisfies a fct'l eqn $s \leftrightarrow 2-s$.

Defn $r_{an}(E) := \operatorname{ord}_{s=1} L(E,s) \in \mathbb{Z}_{\geq 0}$.

BSD conj: $r_{alg}(E) = r_{an}(E)$.

Rmk BSD $\Rightarrow$ an algorithm computing r_{alg}.

Thm A (Gross-Zagier, Kolyvagin 80's)
(1) $r_{an}(E) = 0 \Rightarrow r_{alg}(E) = 0$.
(2) $r_{an}(E) = 1 \Rightarrow r_{alg}(E) = 1$.
(Little is known in higher rank case!).

§3 Bridge: Selmer gps.

$$\begin{tikzcd} E(\mathbb{Q}) \arrow[dr, dash] & & L(E,s) \arrow[dl, dash, "\text{``explicit reciprocity law''}"] \\ & \mathrm{Sel}_{p^\infty}(E) & \end{tikzcd}$$

p^∞-descent

Recall: we have p^n-descent sequence:

$$0 \longrightarrow \frac{E(\mathbb{Q})}{p^n E(\mathbb{Q})} \longrightarrow \underbrace{\mathrm{Sel}_{p^n}(E)}_{\text{finite gp}} \longrightarrow \underbrace{Ш(E)[p^n]}_{\text{Tate-Shafarevich gp.}} \longrightarrow 0$$

Conj. $Ш(E)$ should be finite.

Take inverse limite and $\underset{\mathbb{Z}_p}{\otimes} \mathbb{Q}_p$.

$$E(\mathbb{Q}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_p \hookrightarrow \mathrm{Sel}_{p^\infty}(E)$$

easier to understand.

is an isom. if $Ш(E)$ is finite.

Selmer rk.

So $\underbrace{\text{finiteness of } Ш(E) \Rightarrow r_{alg}(E)}$ $= \dim_{\mathbb{Q}_p} \mathrm{Sel}_{p^\infty}(E)$.

Strategy for Thm A.

(1) $r_{an}(E) = 0 \Rightarrow \mathrm{Selmer}_{p^\infty}(E) = 0.$

(2) $r_{an}(E) = 1 \Rightarrow \dim_{\mathbb{Q}_p} \mathrm{Sel}_{p^\infty}(E) \leq 1$. & $\underbrace{r_{alg}(E) \geq 1}$.

} bd Selmer from above (Kolyvagin) (Euler system).

explicit pt construction (Heegner points) (GZ)

<u>Key in bounding Selmer gp</u>: to construct Galois cohomology classes with <u>controlled local ramification</u>.

There're currently at least 3 ways to do this:

- Kolyvagin: Heegner points in ~~higher~~ ring class fields.
- Bertolini–Darmon: Heegner points on different Shimura curves.

• Kato : Kato classes in cyclotomic fields.
We'll focus on (BD). and generalize to more general motives.

§4. Bloch-Kato conjecture for Rankin-Selberg motives.

Bloch-Kato conj.: $\dim \mathrm{Sel}_{p^\infty}(E) = r_{an}(E)$.
(predicted by BSD).

Generalize: Selmer gp: $H^1_f(\mathbb{Q}, \rho_E)$.
where $\rho_E : G_\mathbb{Q} \longrightarrow GL_2(\mathbb{Q}_p)$ is Galois repn on $T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Picture before:

$$\begin{tikzcd} & f_E \in S_2(N) \arrow[dl] \arrow[dr] & \\ H^1_f(\mathbb{Q}, \rho_E) & & L(f_E, s) \end{tikzcd}$$

$(\simeq \mathrm{Sel}_{p^\infty}(E))$.

Generalization: π : conjugate self-dual cohomological cusp auto repn on $GL_n(\mathbb{A}_K)$ where K/F = CM quad ~~totally real field~~

Chenevier
Harris

$\rho_\pi : G_K \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$.

π on $GL_n(\mathbb{A}_K)$, π' on $GL_{n+1}(\mathbb{A}_K)$.

$$H^1_f(K, \rho_\pi^\vee \otimes \rho_{\pi'}^\vee) \qquad\qquad L(\pi \times \pi', s).$$

Rankin-Selberg motives $\qquad\qquad$ Rankin-Selberg L-function.

BK conj. asserts: $\dim H^1_f(K, \rho_\pi^\vee \otimes \rho_{\pi'}^\vee) = \operatorname{ord}_{s=\frac{1}{2}} L(\pi \times \pi', s)$.

Thm B (LTX~~X~~Z) Assume π, π' has trivial infinitesimal char. and p satisfies certain assumptions:

(1) $\operatorname{ord}_{s=\frac{1}{2}} L(\pi \times \pi', s) = 0 \Rightarrow H^1_f(K, \rho_\pi^\vee \otimes \rho_{\pi'}^\vee) = 0$.

(2) Assume $cl_{GGP}(\pi \times \pi') \in H^1_f(K, \rho_\pi^\vee \otimes \rho_{\pi'}^\vee)$ is nonzero, ~~then~~ Then $\dim H^1_f(K, \rho_\pi^\vee \otimes \rho_{\pi'}^\vee) = 1$.

Rmk. Beilinson conj. $\Rightarrow$ $cl_{GGP}(\pi \times \pi') \neq 0 \Leftrightarrow \operatorname{ord}_{s=\frac{1}{2}} L(\pi \times \pi', s) = 1$.

Defn $\chi(p) = \begin{cases} 1 & p \nmid N \\ 0 & p \mid N. \end{cases}$ then,

$$L_p(E,s) = \left(1 - a_p \cdot p^{-s} + \chi(p)\, p \cdot p^{-2s}\right)^{-1}.$$

Easy consequence: (1) $a_1 = 1$.

(2) $a_{mn} = a_m \cdot a_n$ if $(m,n) = 1$.

(3) $a_{p^r} = a_p\, a_{p^{r-1}} - \chi(p) \cdot p \cdot a_{p^{r-2}}$ $(r \geq 2)$.

This exactly matches relations for Hecke operators T_n.

Rmk $L(E,s)$ is an example of a motivic L-fctn., i.e., associated w/ Galois repns coming from Geometry.

~~Look at ℓ-adic Tate module $V_\ell E = T_\ell E \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \circlearrowleft G_{\mathbb{Q}}$.~~

~~$\rightsquigarrow$ 2-dim'l Galois repn $\rho_E : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_\ell E)$~~

$$\mathrm{Aut}(V_\ell E) \cong GL_2(\mathbb{Q}_\ell).$$

Thm $L_p(E,s) = \det\left(1 - p^{-s}\, \mathrm{Frob}_p \mid (V_\ell E)^{I_p}\right)^{-1}.$

where $I_p \subseteq G_{\mathbb{Q}_p}$ is the inertia subgp $(p \neq \ell)$.

Notice $\dim (V_\ell E)^{I_p} = \begin{cases} 2 & p \nmid N \\ 1 & p \parallel N \\ 0 & p^2 \mid N. \end{cases}$

Prop. $L(E,s)$ converges when $Re(s) > \frac{3}{2}$.

pf. $|a_p| \le 2\sqrt{p}$. $\Longrightarrow \left|\frac{a_n}{n^s}\right| = O\left(\frac{1}{n^{Re(s)-\frac{1}{2}}}\right) \Rightarrow$ Proposition

---

Expected properties of $L(E,s)$ (any motivic L-fctns).

① Euler product (✓)

② analytic continuation to all of $\mathbb{C}$.

③ $L(E,s)$ has a functional eqn.

more difficult, and is one of the motivations of Langlands program.

L-fctns of modular forms.

Let $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \equiv 0 \ (N) \right\} \subseteq SL_2(\mathbb{Z})$.

Let $S_k(N)$ = space of cusp forms of level $\Gamma_0(N)$ and wt k.

Recall $f \in S_k(N)$ if ① $f : \mathfrak{h} \to \mathbb{C}$ holomorphic

② $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma_0(N)$.

③ f is holomorphic and vanishes at all cusps.

Defn. $L(f,s) := \sum_{n\geq 1} \frac{a_n}{n^s}$, $a_n = a_n(f)$. where

$$f(\tau) = \sum_{n\geq 1} a_n(f)\cdot q^n.$$

Rmk. In this generality, $L(f,s)$ may not have Euler product.

It does iff f is an eigenform for all T_n's.

Prop. $L(f,s)$ converges when $Re(s) > \frac{k}{2}+1$.

actually $|a_n| = O(n^{\frac{k}{2}})$.

Rmk This is the trivial bound for $E/\mathbb{Q}$ (weight $k=2$).

Rmk If f is eigenform, has a sharper bound

$$|a_p| = O(p^{\frac{k-1}{2}}). \quad \text{(due to Deligne)}.$$

pf. Notice $a_n = \int_0^1 e^{-2\pi i n(x+iy)} f(x+iy)\, dx$.

~~take~~ for any y, take $y = \frac{1}{n}$.

$$|a_n| = e^{2\pi} \left| \int_0^1 f(x+i\tfrac{1}{n})\, dx \right|.$$

On the other hand:

$|f(\tau)|\cdot(\operatorname{im}\tau)^{k/2}$ is invariant under Γ.

f is cusp form so $\downarrow$ is extended to $(\Gamma\backslash\mathfrak{h})^*$. hence bounded : $|f(x+iy)| = O(y^{-\frac{k}{2}})$.

so $y = \frac{1}{n} \Rightarrow |f(x+i\frac{1}{n})| = O(n^{\frac{k}{2}})$.

$\Rightarrow |a_n| = O(n^{\frac{k}{2}})$.

Next goal: ② & ③ for $L(f,s)$.

Proofs of ② & ③.

Starting pt: $L(f,s)$ has an explicitly integral representation. (Mellin trans. of f).

Defn: The Mellin transform of f is

$$g(s) := \int_0^\infty f(i\cdot t)\cdot t^s \cdot \frac{dt}{t}.$$

Prop: $g(s) = (2\pi)^{-s}\Gamma(s)L(f,s)$ $\quad (Re(s) > \frac{k}{2}+1)$.

pf: recall: $\Gamma(s) := \int_0^\infty e^{-t}\cdot t^s \frac{dt}{t}$.

change t by $2\pi n t$ in

$$\Gamma(s) = (2\pi n)^s \int_0^\infty e^{-2\pi n t} t^s \frac{dt}{t}.$$

$$n^{-s} = (2\pi)^s \Gamma(s)^{-1} \int_0^\infty e^{-2\pi n t} t^s \frac{dt}{t}.$$

$$\Rightarrow L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s} = (2\pi)^s \Gamma(s)^{-1} \sum_{n\geq 1} a_n \int_0^\infty e^{-2\pi n t} t^s \frac{dt}{t}.$$

$$= (2\pi)^s \Gamma(s)^{-1} \int_0^\infty \sum_{n\geq 1} a_n e^{-2\pi n t} t^s \frac{dt}{t}.$$

$$= (2\pi)^s \Gamma(s)^{-1} \overbrace{\int_0^\infty f(it)\, t^s \frac{dt}{t}}^{g(s)} \qquad \boxed{\text{Zy}}$$

Defn. The completed L-fctn is

$$\Lambda(f,s) := N^{\frac{s}{2}} \overbrace{(2\pi)^{-s} \Gamma(s) L(f,s)}^{g(s)}$$

Defn Define an operator $W_N : S_k(N) \longrightarrow S_k(N)$.

$$(W_N f)(\tau) = \left(\frac{i}{\sqrt{N}\,\tau}\right)^k f\left(-\frac{1}{N\tau}\right).$$

$\boxed{\text{Check:}}$ $W_N^2 = 1$, W_N preserves $S_k(N)$.

(W_N is known as Atkin-Lehner involution).

Thm (Hecke) Let $f \in S_k(N)$. Assume $W_N f = \varepsilon \cdot f$ $\quad (\varepsilon \in \{\pm 1\})$.

Then, (1) $\Lambda(f,s)$ has analytic continuation to all of $\mathbb{C}$.

(2) Has a functional eqn:

$$\Lambda(f,s) = \underbrace{\varepsilon}_{\text{"sign of functional eqn"}} \cdot \Lambda(f, k-s)$$

Cor. $L(f,s)$ has analytic continuation to all of $\mathbb{C}$.

($N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s)$ doesn't have any zeros).

Rmk. This thm applies to any newform $f \in S_k(N)$.

$$\Longrightarrow \Lambda(f,s) = \pm \Lambda(f, k-s)$$

(some theory of newforms).

pf of Thm: $\Lambda(f,s) = N^{s/2} \int_0^\infty f(it)\, t^s \frac{dt}{t}$.

$$\overset{t \longmapsto \frac{t}{\sqrt{N}}}{=\!=\!=} \int_0^\infty f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}.$$

$$= \underbrace{\int_0^1 f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}}_{(*)} + \overbrace{\int_1^\infty f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t}}^{\text{converges!}}.$$

$$(*) \underset{\text{defn of } W_N}{=} \int_0^1 (W_N f)\left(\frac{i}{\sqrt{N}\,t}\right) t^{s-k}\, dt/t$$

(Check) $$\overset{t \mapsto t^{-1}}{=\!=\!=} \int_1^\infty (W_N f)\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t}.$$

so $$\Lambda(f,s) = \int_1^\infty f\left(\frac{it}{\sqrt{N}}\right) t^s \frac{dt}{t} + \int_1^\infty (W_N f)\left(\frac{it}{\sqrt{N}}\right) t^{k-s} \frac{dt}{t}.$$

Thm follows since $W_N f = \varepsilon \cdot f$.

§1. Rank part of BSD

$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{ord}_{s=1} \Lambda(E,p) = \begin{cases} \text{even} & \text{if } \varepsilon = 1 \\ \text{odd} & \text{if } \varepsilon = -1. \end{cases}$$

$$\Lambda(E,s) = \varepsilon \cdot \Lambda(E, 2-s).$$

Heuristic: plug in $s=1$ into $\prod_p L_p(E,s)$ (doesn't converge!)

$$L_p(E,1) = (1 - a_p \cdot p^{-1} + \chi(p) p^{-1})^{-1}$$

$$= \left(\frac{p - a_p + \chi(p)}{p}\right)^{-1} = \frac{p}{p - a_p + \chi_p(p)} = \begin{cases} \dfrac{p}{p+1-a_p} & p \text{ good} \\ \dfrac{p}{p \pm 1} & p \text{ multiplicative} \\ \dfrac{p}{p} & p \text{ additive} \end{cases} = \frac{p}{|\widetilde{E}(\mathbb{F}_p)|}.$$

$\widetilde{E}$ is the identity component of Néron model of E.

So: $$L(E,1) \text{ "}=\text{" } \prod_p \frac{p}{|\widetilde{E}(\mathbb{F}_p)|}.$$

$L(E,1)$ becomes "more" zero. $\leftarrow$ $|\widetilde{E}(\mathbb{F}_p)|$ "large" for each p

$\Updownarrow$ $\Uparrow$

$r_{an}(E)$ "large". $E(\mathbb{Q})$ has "more" points or $r_{alg}(E)$ "large"

In fact: Birch + Swinnerton-Dyer (1958) did numerical computation for $\{y^2 = x^3 - n^2 x\}$.

They computed $$\prod_{p<X} \frac{|\widetilde{E}(\mathbb{F}_p)|}{p} \text{ looks like } \sim c \cdot (\log X)^r$$

where $r = r_{alg}(E)$.

BSD conj (rk part):

This leads to $\qquad r_{an}(E) = r_{alg}(E)$.

Q: How to compute $r_{an}(E)$? $\quad L^{(r)}(E,1)$?

§2. Compute leading term $L^{(r)}(E,1)$.

Recall: $\Lambda(f,s) = \int_1^\infty f\left(\frac{it}{\sqrt{N}}\right)\left(t^s + \varepsilon t^{k-s}\right)\frac{dt}{t}$ where $W_N(f) = \varepsilon \cdot f$.

Apply this to $f = f_E \in S_2(N)$.

if $r=0$ $(\Rightarrow \varepsilon = 1)$ $\quad \Lambda(\overset{f}{\cancel{E}},1) = 2\int_1^\infty f\left(\frac{it}{\sqrt{N}}\right) dt$.

$$= 2\int_1^\infty \sum_{n\geq 1} a_n e^{-\frac{2\pi n t}{\sqrt{N}}} dt.$$

$$= 2\sum_{n\geq 1} a_n \int_1^\infty e^{-\frac{2\pi n t}{\sqrt{N}}} dt.$$

$$= 2\sum_{n\geq 1} \frac{\sqrt{N}}{2\pi}\cdot\frac{a_n}{n}\cdot e^{\frac{-2\pi n}{\sqrt{N}}}.$$

$$L(f,1) = \frac{2\pi}{\sqrt{N}}\Lambda(f,1) = 2\sum_{n\geq 1}\frac{a_n}{n}\cdot e^{-2\pi n/\sqrt{N}}.$$

~~Res~~ Rmk: $L(E,1)$ "=" $\sum_{n\geq 1}\frac{a_n}{n}$. So ~~$\Sigma$~~ actual $L(E,1)$ is the "weighted sum".

if $r=1$ $(\Rightarrow \varepsilon = -1)$ $\quad \Lambda'(f,1) = 2\int_1^\infty f\left(\frac{it}{\sqrt{N}}\right)\log t \, dt$.

$$= 2\int_1^\infty \sum_{n\geq 1} a_n e^{-\frac{2\pi n t}{\sqrt{N}}}\log t \, dt.$$

$$= 2\sum_{n\geq 1} a_n \int_1^\infty \cancel{a_n}\, e^{\frac{-2\pi n t}{\sqrt{N}}}\log t \, dt.$$

$$= 2\sum_{n\geq 1}\frac{\sqrt{N}}{2\pi}\frac{a_n}{n}\int_1^\infty e^{-\frac{2\pi i t}{\sqrt{N}}}\frac{dt}{t}$$

$$\Longrightarrow L'(f,1) = \frac{2\pi}{\sqrt{N}}\Lambda'(f,1) = 2\sum_{n\geq 1}\frac{a_n}{n}\int_1^\infty e^{-\frac{2\pi n t}{\sqrt{N}}}\frac{dt}{t}.$$

More generally, for any $r \geq 1$ define

$$E_r(x) \stackrel{\Delta}{=} \int_1^\infty e^{-xt} (\log t)^{r-1} \frac{dt}{t} \cdot \frac{1}{(r-1)!}.$$

Then the leading term: $L^{(r)}(f,1) = 2r! \sum_{n\geq 1} \frac{a_n}{n} E_r\left(\frac{2\pi n}{\sqrt{N}}\right).$

Rmk: ε is easy to compute:

Break the integral into $\int_A^\infty + \int_{A^{-1}}^\infty$

$$\Rightarrow L(f,1) = 2 \sum_{n\geq 1} \frac{a_n}{n}\left(e^{-\frac{2\pi n}{A\sqrt{N}}} + \varepsilon e^{-\frac{2\pi n A}{\sqrt{N}}}\right).$$

plug in different A's ($A=1.0$, $A=1.01$) gives the only choice for ε.

---

§ 3 $r=0$: $L(E,1) = ?$

Example: $E: y^2 = x^3 - x$ $N=32$.

$$f = q \prod_{n\geq 1} (1-q^{4n})^2 (1-q^{8n})^2.$$

$$= q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} + \cdots$$

$(a_p \neq 0 \Leftrightarrow p \equiv 1 \ (4))$.

$$L(E,1) = 2\sum_{n\geq 1} \frac{a_n}{n} e^{-\frac{2\pi n}{\sqrt{32}}} = 0.6555143883 7\cdots$$

Example: $E: y^2 = x^3 + 1$ $N=36$.

$$f = q \prod_{n\geq 1} (1-q^{6n})^4.$$

$$= q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - \cdots$$

$(a_p \neq 0 \Leftrightarrow p \equiv 1 \ (3))$.

$$L(E,1) = 2\sum_{n\geq 1} \frac{a_n}{n} e^{-\frac{\pi n}{3}} \doteq 0.7010910\,52663\cdots$$

Q: What are these transcendental numbers?

A: natural transcendental numbers ass. to E (periods of elliptic curve)

Defn: Suppose E has minimal Weierstrass eqn:

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

define Néron differential

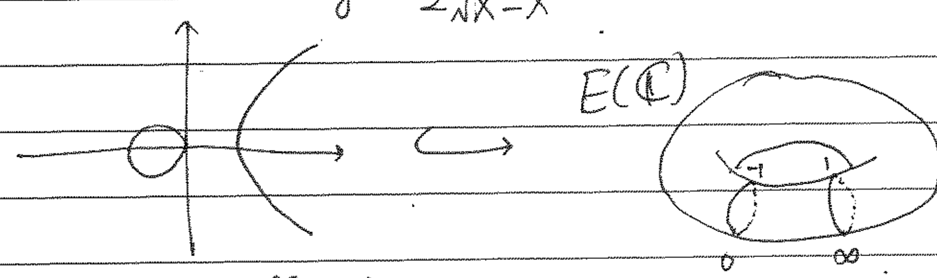
$$\omega_E := \frac{dx}{2y + a_1 x + a_3} \in H^0(E, \Omega^1_E).$$

Néron period is:

$$\Omega(E) = \int_{E(\mathbb{R})} \omega_E \in \mathbb{R}.$$

Ex: $E: y^2 = x^3 - x$. $\quad \omega_E = \frac{dx}{2y} = \frac{dx}{2\sqrt{x^3-x}}$.

$E(\mathbb{R})$:

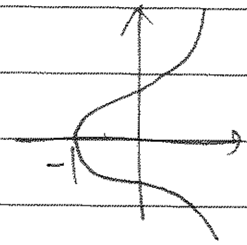


$$\Omega(E) = \int_{E(\mathbb{R})} \omega_E = 4\int_1^\infty \frac{dx}{2\sqrt{x^3-x}} = 2\int_1^\infty \frac{dx}{\sqrt{x^3-x}} \doteq 5.244115108\cdots$$

Observe: $\frac{L(E,1)}{\Omega(E)} \approx \frac{1}{8} \in \mathbb{Q}$.

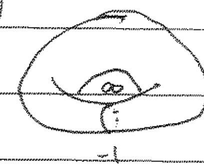
Ex. $E: y^2 = x^3+1$ $\qquad \omega_E = \dfrac{dx}{2\sqrt{x^3+1}}$.

$E(\mathbb{R})$



$\hookrightarrow$

$E(\mathbb{C})$



so: $$\Omega(E) = \int_{-1}^{\infty} \frac{dx}{\sqrt{x^3+1}} = 4.2065463|\ldots$$

$$\Rightarrow \frac{L(E,1)}{\Omega(E)} \approx \frac{1}{6} \in \mathbb{Q}.$$

Conj: $\dfrac{L(E,1)}{\Omega(E)} \in \mathbb{Q}$ for $E/\mathbb{Q}$.

§1. $L(E,1)/\Omega(E) \in \mathbb{Q}$.

~~$\int_{E(\mathbb{R})} \omega_E$ = period of Weierstrass $\wp$-function (inverse of elliptic integration).~~

Goal $\quad \overset{= L(f_E,1)}{L(E,1)}$ is related to periods of $X_0(N) = \Gamma_0(N) \backslash \mathcal{H}^*$

Recall: $\quad$ Mellin transform ~~of~~ interpretation of $L(f,s)$:

$$\cancel{L(f,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty f(it)\, t^s \frac{dt}{t}}.$$

apply to "$s=1$": $\quad L(f,1) = \frac{2\pi}{1} \cdot \int_0^\infty f(it)\, dt \overset{\tau = it}{=\!=\!=} -2\pi i \int_0^{i\infty} f(\tau)\, d\tau.$

$\omega_f = 2\pi i f(\tau) d\tau$ is a 1-form on $X_0(N)$.

$$\cancel{L(f,1) = -\int_0^{i\infty} \omega_f}.$$

Thm (Manin-Drinfeld) $\quad$ Take any cusps $\alpha, \beta \in X_0(N)$. Then $\alpha \longrightarrow \beta$ belongs to

$$\cancel{H_1(X_0(N), \mathbb{Q})}.$$

i.e. $\exists\ \gamma_i \in H_1(X_0(N), \mathbb{Z})$ and $c_i \in \mathbb{Q}$, s.t.

$$\cancel{\int_\alpha^\beta \omega_f = \sum c_i \int_{\gamma_i} \omega_f.} \quad \cancel{\forall f \in S_2(N)}.$$

In other words, a rational multiple of $L(f,1)$ is indeed a multiple.

Rmk. $\quad \alpha - \beta$ gives a torsion point on $J_0(N) = \mathrm{Jac}(X_0(N))$.

Idea of pf: $\quad$ use Hecke operator T_p for $p \nmid N$. T_p acts by scalar on ~~modular~~ eigen forms.

(?).

Thm. (Birch) $\dfrac{L(E,1)}{\Omega(E)} \in \mathbb{Q}$

pf. The modularity theorem gives a nontrivial map $X_0(N) \xrightarrow{\varphi} E$ s.t. $\varphi^*(\omega_E) = c \cdot 2\pi i f(\tau)\, d\tau$, $c \in \mathbb{Q}^\times$

conj. $c = 1$

---

§2. What is $\dfrac{L^{(r)}(E,1)}{\Omega(E)} = ?$ $r = r_{an}(E)$.

Ex: $E: y^2 = x^3 - 25x$, $N = 32 \cdot 25 = 800$.

$r = 1$, $E(\mathbb{Q}) \cong \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/2$.

$(-4,6)$ $(0,0)$ $(5,0)$

$f = q - 3q^9 - 6q^{13} - 2q^{17} + \cdots$

We can use formula last time:

$$L'(f,1) = 2 \sum \frac{a_n}{n} E\left(\frac{2\pi n}{\sqrt{800}}\right) = 2.2273703795 4\ldots$$

$$\Omega(E) = 2 \int_5^\infty \frac{1}{\sqrt{x^3 - 25x}} = 2.34523957\ldots$$

$$\frac{L'(f,1)}{\Omega(E)} = 0.949741\ldots \notin \mathbb{Q}\,!$$

Idea: use arithmetic complexity of the rat'l point of infinite order.

Defn. for $x = \frac{a}{b} \in \mathbb{Q}^\times$, $(a,b) = 1$, define its height by $h(x) = \log \max\{|a|, |b|\}$.

Defn. $p \in E(\mathbb{Q})$, define its naive height $h(p) := h(x(p))$.

Problem: this depends on Weierstrass eqn.

Tate's canonical height. (or Néron-Tate height)

Refn $\hat{h}(P) := \lim_{n\to\infty} \frac{h([2^n]\cdot P)}{4^n} \in \mathbb{R}_{\geq 0}$.

"or $= \lim_{N\to\infty} \frac{h([N]P)}{N^2}$"

Example: $P = (-4, 6)$ $\quad h(P) = 1.856786\cdots$

$2P = \left(\frac{1681}{144}, \cdots\right)$ $\quad \frac{h(2P)}{4} = 1.8778407\ldots$

$4P = \left(\frac{1118341279392}{2234161324161}, \cdots\right)$, $\quad \frac{h(4P)}{16} = 1.89946579\ldots$

$\hat{h}(P) = 1.8994821725\ldots$

Observe: $\frac{L'(E,1)}{\Omega(E)\,\hat{h}(P)} = \frac{1}{2} \in \mathbb{Q}$!

Rmk: $\hat{h}(P)$ doesn't depend on Weierstrass equation and is uniquely characterized by the following:

(1) $\hat{h} : E(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0}$.

(2) $\hat{h}(2P) = 4\hat{h}(P)$.

(3) $\hat{h}(P) - h(P)$ is bounded for $P \in E(\mathbb{Q})$.

Defn. (why linear?) ($r \geq 2$) define Néron-Tate height pairing.

$\langle -, - \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0}$.

$\langle P, Q \rangle := \frac{1}{2}\left(\hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)\right)$.

In particular $\langle P, P \rangle = \hat{h}(P)$

$\{P_i\}_{i=1}^{r}$ generate $\frac{E(\mathbb{Q})}{E(\mathbb{Q})_{tors}}$

Defn. The regulator is defined by $R(E) \overset{\det}{=} (\langle P_i, P_j \rangle)_{r\times r}$.

Rmk. (1) $\hat{h}(P)=0 \Leftrightarrow P\in E(\mathbb{Q})_{tors}$.

~~(why?)~~ so $\langle -,-\rangle$ is ~~perfect on $\frac{E(\mathbb{Q})}{E(\mathbb{Q})_{tors}} \Rightarrow R(E)\neq 0$.~~

(2) $r=0$ $R(E):=1$.

$r=1$ $R(E)=\hat{h}(P)$.

Conj. $\frac{L^{(r)}(E,1)}{\Omega(E)R(E)}\in\mathbb{Q}^\times$.

Rmk: $r=0$, it is proved

$r=1$, it is true (by $r=0$ + Gross-Zagier formula).

~~$r\geq 2$, no example is known!~~

(enormous numerical ~~examples~~ evidences)

---

§3. Full BSD conj.

Goal: Find a formula for $\frac{L^{(r)}(E,1)}{\Omega(E)R(E)}$. $\left(\prod c_p\right)$

Ex:

E	N	r			
$y^2=x^3-x$	32	0	$\frac{1}{8}$	$c_2=2$	$\frac{1}{16}$
$y^2=x^3+1$	36	0	$\frac{1}{6}$	$c_2c_3=3\cdot 2=6$	$\frac{1}{36}$
$y^2=x^3-25x$	800	1	$\frac{1}{2}$.	$c_2\cdot c_5=2\cdot 4=8$.	$\frac{1}{16}$

Idea: We looked at $\int_{E(\mathbb{R})}\omega_E$. Should look at $\int_{E(\mathbb{Q}_p)}|\omega_E|$

Fact: $\int_{E(\mathbb{Q}_p)}|\omega_E| = [E(\mathbb{Q}_p):E^\circ(\mathbb{Q}_p)]\frac{|\tilde{E}(\mathbb{F}_p)|}{p}$ identity component of Néron model
smooth part

where $E^\circ(\mathbb{Q}_p)=\{P\in E(\mathbb{Q}_p): P \bmod p\in\tilde{E}(\mathbb{F}_p)\}$

Defn. The local Tamagawa #: $c_p(E) := [E(\mathbb{Q}_p) : E^0(\mathbb{Q}_p)]$.
$(= |\text{component gp}|)$.

~~Guess~~
Guess: $\dfrac{L^{(r)}(E,1)}{\Omega(E)\prod_p c_p R(E)} = $ ~~torsion gp~~ $|E(\mathbb{Q})_{tor}|^{-2}$.

each example: $\mathbb{Z}/2 \times \mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/3$, $\mathbb{Z}/2 \times \mathbb{Z}/2$.

---

§1. Full BSD.

Last time: we saw few examples:

$$\frac{L^{(r)}(E,1)}{\Omega(E)\prod_p c_p(E) R(E)} \overset{?}{=} \frac{1}{|E(\mathbb{Q})_{tors}|^2} \qquad (r=0,1)$$

Rewrite: $$\frac{L^{(r)}(E,1)}{r!} \overset{?}{=} \underbrace{\Omega(E)\prod_p c_p(E)}_{\text{period}} \cdot \frac{\overbrace{R(E)}^{\text{regulator}}}{|E(\mathbb{Q})_{tors}|^2}.$$

Rmk: $\dfrac{R(E)}{|E(\mathbb{Q})_{tors}|^2}$ is more canonical: choose any $\overset{\text{lattice}}{\Lambda} \subseteq E(\mathbb{Q})$ of rk r. $\dfrac{R(\Lambda)}{[E(\mathbb{Q}):\Lambda]^2}$ is independent of choice of Λ.

Notice: the similarity w/ class number formula:
Let $K/\mathbb{Q}$ number field.

$$\operatorname{res}_{s=1} \zeta_K(s) = \underbrace{\frac{2^{r_1}(2\pi)^{r_2}}{|d_K|^{1/2}}}_{\text{period}} \cdot \underbrace{\frac{R(K)}{|\mathcal{O}^{\times}_{K,tor}|}}_{\text{torsion}} \cdot \underbrace{h(K)}_{\text{class \#}}.$$

Missing in BSD: an analogue of $h(K) = |Cl(K)|$. It's provided by $|Ш(E)|$.

Full BSD conj: $E/\mathbb{Q}$ elliptic curve.

Thm. (1) rank part: $r_{alg}(E) = r_{an}(E)$.

(2) formula part:

$$\frac{L^{(r)}(E,1)}{r!} = \Omega(E) \cdot \prod_p c_p(E) \frac{R(E)}{|E(\mathbb{Q})_{tors}|^2} |Ш(E)|.$$

Rmk. (1) is known when $r_{an}(E) \leq 1$. (G-Z, K).

(2) is known ~~when~~ for many $E/\mathbb{Q}$ with $r_{an}(E) \leq 1$ (but not all!).

---

§2. $Ш(E)$.

Defn. The Tate-Shafarevich gp of E is defined to be

$$Ш(E) := \ker\left(H^1(\mathbb{Q}, E) \longrightarrow \prod_v H^1(\mathbb{Q}_v, E)\right).$$

In other words: an element of $Ш(E)$ is represented by a genus 1 curve $C/\mathbb{Q}$ s.t. $\mathrm{Jac}(C) \simeq E$ and $C(\mathbb{Q}_v) \neq \phi$ for all v, places of $\mathbb{Q}$.

So $Ш(E)$ measures failure of local-to-global ~~principal~~ principle for $\mathbb{Q}$-points (for C).

Example: (Lind 1940) He found $C: 2y^2 = x^4 - 17$ has points over all $\mathbb{Q}_p$ and $\mathbb{R}$. but has no $\mathbb{Q}$-point.

This corresponds to $E = \mathrm{Jac}: y^2 = x^3 + 17x$. $(r=0)$

has $Ш(E) \simeq (\mathbb{Z}/2)^2$.

Check BSD:

$L(E,1) = 3.6523\cdots$

$\Omega(E) = 1.82618\ldots$

$\prod_p c_p(E) = c_2 \cdot c_{17} = 1 \cdot 2 = 2.$

$E(\mathbb{Q})_{tors} = \mathbb{Z}/2\mathbb{Z}.$

$$\frac{L(E,1)}{\Omega(E)\prod_p c_p} = \frac{2}{1\cdot 2} = 1.$$

$$\frac{|Ш(E)|}{|E(\mathbb{Q})_{tors}|^2} = \frac{4}{2^2} = 1.$$

Example (Selmer, 1951) He found, $C: 3X^3+4Y^3+5Z^3=0$. has points over all $\mathbb{Q}_p$ and $\mathbb{R}$ but not $\mathbb{Q}$-point.

$$J(C)=E: \quad X^3+Y^3+60Z^3=0.$$

or equivalently: $\quad y^2=x^3-24300$

has $Ш(E)=(\mathbb{Z}/3)^2$.

Rmk: In general, no algorithm to compute $Ш(E)$. Full BSD helps to compute $|Ш(E)|$.

Rmk: The class gp $Cl(K)$ can also be described in terms of Galois cohomology: $S=\mathrm{Res}_{O_K/\mathbb{Z}}\mathbb{G}_m$.

$$Cl(K)\cong \mathrm{Ker}\left(H^1(\mathbb{Q},S)\longrightarrow \prod_v H^1(\mathbb{Q}_v,S)\right).$$

measuring failure of any ideal that's locally principal but not globally principal

We know finiteness for $Cl(K)$ (using geometry of numbers), but we don't know for $Ш(E)$.

Conj. $Ш(E)$ is finite.

Rmk: (1) Conj. is true if $r_{an}(E)\le 1$ (G-Z; K).

(2) No example when $r_{an}(E)\ge 2$!

(3) Cassels-Tate pairing:

$\exists$ a nondegenerate alternating bilinear form

$$Ш(E)\times Ш(E)\longrightarrow \mathbb{Q}/\mathbb{Z}.$$

whose kernel is divisible part of $Ш(E)$.

(4) Delaunay (2001) developed Cohen-Lenstra heuristics for $Ш(E)$ as ...

(5) It's known that for $p = 2, 3, 5, 7, 13$ ($X_0(p)$ has genus 0).

$Ш(E)[p^\infty]$ can be arbitrarily large.

---

§3. Reformulation of BSD formula (Bloch).

in terms of Tamagawa numbers

Defn Let G be a semisimple algebraic gp/$\mathbb{Q}$. Take top differential ω (defined over $\mathbb{Q}$) of G.

Then $|\omega|_v$ gives a Haar measure μ_v on $G(\mathbb{Q}_v)$.

Then <u>Tamagawa measure</u> is $\mu = \prod_v \mu_v$ on $G(\mathbb{A})$

The Tamagawa number of G is

$$\tau(G) := \mu\left(G(\mathbb{Q}) \backslash G(\mathbb{A})\right).$$

Weil's Conj: If G is simply conn'd, then $\tau(G) = 1$. (This is proved by Kottwitz, Langlands, Lai ⋯)

Example: $G = SL_2/\mathbb{Q}$. $\tau(G) = \mu_\infty\left(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})\right) \cdot \prod_p \mu_p(SL_2(\mathbb{Z}_p))$.

But $\mu_p(SL_2(\mathbb{Z}_p)) = \dfrac{|SL_2(\mathbb{F}_p)|}{p^3} = (1 - p^{-2}) = \zeta_p(2)^{-1}$

So $\tau(G) = 1 \Rightarrow \mu_\infty\left(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})\right) = \zeta(2) = \dfrac{\pi^2}{6}$

$\left(\Longrightarrow \mathrm{Vol}\left(SL_2(\mathbb{Z}) \backslash \mathcal{H}\right) = \dfrac{\pi}{3}\right)$.

Example: $G = SL_n/\mathbb{Q}$. $\tau(G) = 1 \Leftrightarrow \mu_\infty\left(SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R})\right) = \zeta(2) \cdots \zeta(n)$

$\Leftrightarrow \mathrm{Vol}\left(SL_n(\mathbb{Z}) \backslash \mathcal{H}^n\right) = n 2^{n-1} \dfrac{\zeta(2) \cdots \zeta(n)}{\mathrm{Vol}(S^1) \cdots \mathrm{Vol}(S^{n-1})}$.

We can do this for E too (next time).

§1. Continuation of Bloch's reformulation

But for more general alg. gps, $\tau(G)$ may not be well defined.

Ex. $G = \mathbb{G}_m$. $\mu_p(\mathbb{G}_m(\mathbb{Z}_p)) = \frac{|\mathbb{G}_m(\mathbb{F}_p)|}{p} = 1 - \frac{1}{p}$.

Notice: $\prod_p (1-\frac{1}{p})$ doesn't converge!

Idea: regularize this product by changing the local measure to be

$$(1-\frac{1}{p})^{-1}\mu_p = \zeta_p(1)\cdot\mu_p.$$

Take $\frac{\mu(G(\mathbb{Q})\backslash G(\mathbb{A}))}{\zeta^*(1)}$ ← leading coefficient of $\zeta(s)$ at $s=1$.

$\rightsquigarrow \tau(\mathbb{G}_m) = 1$.

For a general linear alg. gp, $\exists\ L(G,S)$ s.t. $\forall p \notin S$ ← finite set of bad primes

$$\frac{|G(\mathbb{F}_p)|}{p^{\dim G}} = L_p(G,1)^{-1}.$$

Ex. $G = GL_n$, $L(G,s) = \zeta(s)\cdot\zeta(s+1)\cdot\dots\cdot\zeta(s+n-1)$.

Defn. For any algebraic gp G, define the modified Tamagawa measure

$$\mu^* = \prod_{v\notin S} L_v(G,1)\,\mu_v \prod_{v\in S}\mu_v.$$

define $\tau(G) = \frac{\mu^*(G(\mathbb{Q})\backslash G(\mathbb{A}))}{L_S^*(G,1)}$

Thm (Generalized Weil's conj.) For G conn'd linear alg gp$/\mathbb{Q}$:

$$\tau(G) = \frac{|\mathrm{Pic}(G)_{tor}|}{|Ш(G)|}.$$

~~E~~ **Rmk**
- G linear, $Ш(G)$ is known to be finite (Borel-Serre '66).
- G simply conn'd + semisimple, $|\mathrm{Pic}(G)_{tors}| = 1$ and $|Ш(G)| = 1$ (hard!)

Bloch Look at elliptic curves. Suppose $r_{alg}(E) = r$. Take $P_i \in E(\mathbb{Q})$ generators.
Use $E \simeq \hat{E} = \mathrm{Ext}^1(E, \mathbb{G}_m) \overset{P_i}{\rightsquigarrow}$ ~~Ext~~ $0 \to \mathbb{G}_m \to X_{P_i} \to E \to 0$.
$\{P_i\}$ basis of $E(\mathbb{Q}) \rightsquigarrow 0 \to \mathbb{G}_m^r \to X \to E \to 0$.

~~(Bloch)~~ **Thm.** ~~Conj.~~ The Tamagawa number conj. for $X \Leftrightarrow$ BSD formula for E.
(in fact, $Ш(X) = Ш(E)$, and Bloch's thm gives an interpretation of $R(E)$ also as a volume!)

---

§2. p-Selmer gp.

Start w/ $0 \to E[p] \to E \xrightarrow{p} E \to 0$.

Take $H^*(\mathbb{Q}, -)$ get a long exact sequence:

$$0 \to E(\mathbb{Q})[p] \to E(\mathbb{Q}) \xrightarrow{\cdot p} E(\mathbb{Q}) \to H^1(\mathbb{Q}, E[p]) \to H^1(\mathbb{Q}, E) \xrightarrow{\cdot p} H^1(\mathbb{Q}, E)$$

$$\begin{tikzcd}
0 \arrow[r] & E(\mathbb{Q})/pE(\mathbb{Q}) \arrow[r, "\delta"] \arrow[d] & H^1(\mathbb{Q}, E[p]) \arrow[r] \arrow[d, "loc"] & H^1(\mathbb{Q}, E)[p] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_v \frac{E(\mathbb{Q}_v)}{pE(\mathbb{Q}_v)} \arrow[r, "\delta_v"] & \prod_v H^1(\mathbb{Q}_v, E[p]) \arrow[r] & \prod H^1(\mathbb{Q}_v, E)[p] \arrow[r] & 0
\end{tikzcd}$$

infinite dim'l $\mathbb{F}_p$-space

Defn. $\mathrm{Sel}_p(E) := \{ x \in H^1(\mathbb{Q}, E[p]) : \mathrm{loc}_v(x) \in \mathrm{im}(\delta_v) \ \forall v \}$.

By construction, we have "p-descent exact seq."

$$0 \to \frac{E(\mathbb{Q})}{pE(\mathbb{Q})} \to \mathrm{Sel}_p(E) \to Ш(E)[p] \to 0.$$

Thm: $\mathrm{Sel}_p(E)$ is finite.

Idea of pf: $\mathrm{Sel}_p(E) \subseteq H^1(\mathbb{Q}, E[p])$ can be thought of as a subspace "cut out" by local conditions $\mathrm{im}(\delta_v) \subseteq H^1(\mathbb{Q}_v, E[p])$ $(\forall v)$.

Cor (p-descent)

$$\dim_{\mathbb{F}_p} \mathrm{Sel}_p(E) = \underbrace{\dim_{\mathbb{F}_p} \frac{E(\mathbb{Q})}{pE(\mathbb{Q})}}_{r_{alg}(E) + \dim E(\mathbb{Q})[p]} + \dim_{\mathbb{F}_p} Ш(E)[p].$$

Rmk $\mathrm{Sel}_p(E)$ is most computable when $p=2$.

Example (Heath-Brown, '94) $\{y^2 = x^3 - n^2 x\}$. He proved the probability $(\dim \mathrm{Sel}_2(E) = d + \overset{E(\mathbb{Q})[2]}{\overbrace{2}})$

$$p(d) = \prod_{k \geq 0} \left(1 + \frac{1}{2^k}\right)^{-1} \cdot \prod_{k=1}^{d} \frac{2}{2^k - 1}.$$

d	$p(d)$
0	0.2097
1	0.4194
2	0.2796
3	0.0799
4	0.0107

Rmk. Poonen-Rains developped Cohen-Lenstra heuristic (for all $E/\mathbb{Q}$).

$$\mathrm{Prob}(\dim \mathrm{Sel}_p = d) = \prod_{k \geq 0} \left(1 + \frac{1}{p^k}\right)^{-1} \prod_{k=1}^{d} \left(\frac{p}{p^k - 1}\right).$$

$\Rightarrow$ (1) Average$\left(|Sel_p(E)|\right) = p+1$.

(proved by Bhargava–Shankar for $p=2,3$, and 5).

(2) $\text{prob}(r_{alg}(E) \geq 2) \leq \frac{1}{p} + \frac{1}{p^2} \Rightarrow$ ~~prob~~ 0% of E has $r_{alg} \geq 2$

Rmk. (1) $Sel_p(E) = 0 \Rightarrow r_{alg}(E) = 0$.

(2) $\dim Sel_p(E) = 1, \forall p \Longrightarrow$ if $Ш(E)[p^\infty] < \infty$ $r_{alg}(E) = 1$.

This is still open in general, but Skinner–Zhang proved many cases, and combining Bhargava, get $\geq 20\%$ of $E/\mathbb{Q}$ has $r_{alg} = 1$.

---

§1. p^∞-Selmer gp.

$$0 \to \frac{E(\mathbb{Q})}{p^n E(\mathbb{Q})} \to Sel_{p^n}(E) \to Ш(E)[p^n] \to 0.$$

take $\underset{n}{\text{colim}}$, get $0 \to E(\mathbb{Q}) \underset{\mathbb{Z}}{\otimes} \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \to Sel_{p^\infty}(E) \to Ш(E)[p^\infty] \to 0$.

Defn. $Sel_{p^\infty}(E) := \underset{n}{\text{colim}}\, Sel_{p^n}(E)$

Rmk: $\cdot\ \exists$ exact sequence

$$0 \to \frac{E(\mathbb{Q})[p^\infty]}{p^n E(\mathbb{Q})[p^\infty]} \to Sel_{p^n}(E) \to Sel_{p^\infty}(E)[p^n] \to 0.$$

$\cdot$ In particular, if $E(\mathbb{Q})[p] = 0$, then $Sel_{p^n}(E) = Sel_{p^\infty}(E)[p^n]$.

Notice: $Sel_{p^\infty}(E) \xrightarrow{\sim} \left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right)^r \oplus$ finite gp.

Defn. $r_p(E) :=$ corank of $Sel_{p^\infty}(E)$, i.e., number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$.

Cor. (1) $r_p(E) \geq r_{alg}(E)$.

(2) $r_p(E) = r_{alg}(E) \Leftrightarrow Ш(E)[p^\infty]$ is finite.

In particular, BSD + finiteness of Ш implies

Conj. (Bloch-Kato). $r_p(E) = r_{an}(E)$. $\forall p$.

Rmk: · $r_p(E)$ is more accessible than $r_{alg}(E)$ due to its local nature.

· $Sel_{p^\infty}(E)$ can be defined in terms of $E[p^\infty]$ or $V_p(E)$. (Thm of BK).

Warning: in general, $Sel_p(E)$ may not depend on $E[p]$!

Target Thm (Gross-Zagier, Kolyvagin) (1) $r_{an}(E)=0 \Longrightarrow r_p(E)=0$.

(2) $r_{an}(E)=1 \Longrightarrow r_p(E) \leq 1$. & $r_{alg}(E) \geq 1$

Today's Goal: Illustrate how to bound $Sel_{p^\infty}(E)$ from above in the simplest case.

§2. Tate's local duality.

(allow us to quantify different local conditions in $H^1(\mathbb{Q}_v, E[p^n])$).

Defn. Let M be a finite G_K-module (e.g. $M = E[p^n]$)

$M^\vee(1) := \mathrm{Hom}(M, \mathbb{G}_m)$. (~~twiste~~ Cartier dual).

Get a natural pairing $M \times M^\vee(1) \longrightarrow \mathbb{G}_m$.

together with cup product we have a pairing

$$H^i(\mathbb{Q}_v, M) \times H^{2-i}(\mathbb{Q}_v, M^\vee(1)) \xrightarrow{\langle\cdot,\cdot\rangle_v} H^2(\mathbb{Q}_v, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

Thm (Tate) (1) $\langle\cdot,\cdot\rangle_v$ is perfect.

(2) (Euler characteristic): $\dfrac{|H^1(\mathbb{Q}_v, M)|}{|H^0(\mathbb{Q}_v, M)|\cdot|H^2(\mathbb{Q}_v, M)|} = |M \otimes \mathbb{Z}_v|$

Apply to $M = E[p^n]$, $M^\vee(1) \cong E[p^n]$.

Cor. We have an isom.: $H^0(\mathbb{Q}_v, E[p^n]) \cong H^2(\mathbb{Q}_v, E[p^n])^\vee$.

and $H^1(\mathbb{Q}_v, E[p^n]) \times H^1(\mathbb{Q}_v, E[p^n]) \longrightarrow \mathbb{Q}/\mathbb{Z}$, perfect.

Cor. when $v \neq p$, then get

$$|H^1(\mathbb{Q}_v, E[p^n])| = |H^0(\mathbb{Q}_v, E[p^n])|^2 = |E(\mathbb{Q}_v)[p^n]|^2.$$

Ex. $n=1$ $\dim_{\mathbb{F}_p} H^1(\mathbb{Q}_v, E[p]) = 2 \dim_{\mathbb{F}_p} E(\mathbb{Q}_v)[p]$.

Defn. Let $\mathcal{L}_v \subseteq H^1(\mathbb{Q}_v, E[p^n])$ be the image of

$$\frac{E(\mathbb{Q}_v)}{p^n E(\mathbb{Q}_v)} \xrightarrow{\delta_v} H^1(\mathbb{Q}_v, E[p^n]).$$

Then $Sel_{p^n}(E) = \{x \in H^1(\mathbb{Q}, E[p^n]) : loc_v(x) \in \mathcal{L}_v, \forall v\}$.

Thm: (Tate). $\mathcal{L}_v$ is its own annihilator under $\langle \cdot, \cdot \rangle_v$.

§3. local conditions at good primes.

Want: pin down $\mathcal{L}_v$ in terms of only $E[p^n]$.

① $v \neq p$.

Defn ("unramified condition") Assume $|M|$ is invertible in $\mathbb{Z}_v$. Defin $H^1_f(\mathbb{Q}_v, M) := H^1(\mathbb{F}_v, M^{I_v}) \subseteq H^1(\mathbb{Q}_v, M)$.

$$0 \to H^1(\mathbb{F}_v, M^{I_v}) \to H^1(\mathbb{Q}_v, M) \to H^1(I_v, M)^{\mathrm{Frob}_v = 1} \to 0$$

Defn. ("singular condition") $H^1_{sing}(\mathbb{Q}_v, M) = H^1(I_v, M)^{\mathrm{Frob}_v=1} \left(= \frac{H^1(\mathbb{Q}_v, M)}{H^1_f(\mathbb{Q}_v, M)} \right)$.

Prop. $H^1_f(\mathbb{Q}_v, M)$ and $H^1_f(\mathbb{Q}_v, M^\vee(1))$ are annihilators of each other.

Cor. $H^1_f(\mathbb{Q}_v, M) \times H^1_{sing}(\mathbb{Q}_v, M^\vee(1)) \to \mathbb{Q}/\mathbb{Z}$ is a perfect pairing.

Apply to $M = E[p^n]$.

Prop. If $v \neq p$. and E has good reduction at v.
Then $\mathcal{L}_v = H^1_f(\mathbb{Q}_v, E[p^n]) \left(= H^1_{\acute{e}t}(\mathbb{Z}_v, \mathcal{E}[p^n])\right)$.

$$H^1(\mathbb{Q}_p, M) \\ \cup\!\!\mid$$

(2) $v = p$.

Defn. (flat condition) $M/\mathbb{Z}_p$ finite flat gp scheme. Define $H^1_f(\mathbb{Q}_p, M) := H^1_{fppf}(\mathbb{Z}_p, M)$

Prop. $H^1_f(\mathbb{Q}_p, M)$ and $H^1_f(\mathbb{Q}_p, M^\vee(1))$ are annihilators of each other.

Prop. If $v = p$, E has good reduction at p, then
$$\mathcal{L}_p = H^1_f(\mathbb{Q}_p, \mathcal{E}[p^n]).$$

Rmk. $n=1$, $p>2$, Raynaud shows $E[p]$ has a unique extn to $\mathbb{Z}_p$.
So $H^1_f(\mathbb{Q}_p, \mathcal{E}[p])$ is determined by $E[p]$.

---

§4. Kolyvagin's method.

Starting pt: global class field theory: for $s_1, s_2 \in H^1(\mathbb{Q}, E[p])$.
we have $\sum_v \langle s_1, s_2 \rangle_v = 0$.

Thm Let $S = \{\text{bad primes for } E\} \cup \{p\}$. Assume we can construct $C(\ell) \in H^1(\mathbb{Q}, E[p])$ for $\ell \notin S$.
satisfying (1) $v \notin S \cup \{\ell\}$
$\mathrm{loc}_v(C(\ell)) \in H^1_f(\mathbb{Q}_v, E[p])$.
(2) $v = p$ $\mathrm{loc}_p(C(\ell)) \in H^1_f(\mathbb{Q}_p, E[p])$.
(3) $v \in S - \{p\}$. $\mathrm{loc}_v(c(\ell)) = 0$. (e.g. $E(\mathbb{Q}_v)[p] = 0$).
(4) $v = \ell$. $H^1_f(\mathbb{Q}_v, E[p]) = \mathbb{F}_p$ and $\mathrm{loc}_\ell(c(\ell)) \neq 0$.
$$\mathrm{loc}_\ell(c(\ell)) \in H^1_{sing}(\mathbb{Q}_\ell, E[p])$$

Thm. $\mathrm{Sel}_p(E) = 0$.

Last time (I missed!)

K satisfies Heegner hypothesis for $X_0(N)$ if
(im. quad.)

$$p \mid N \Rightarrow p \text{ splits in } K.$$

$\rightsquigarrow$ Heegner pts $x_K \in X_0(N)(H_K) \rightsquigarrow y_K \in E(K)$.

~~Let E_K be the base change of E from $\mathbb{Q}$ to K.~~

$\rightsquigarrow$ $L(E_K, s)$ satisfies a functional eqn.

$$L(E_K, s) \longleftrightarrow L(E_K, 2-s)$$

with sign of functional eqn $\varepsilon(E_K)$

Prop. Assume $(d_K, N) = 1$. Then $\varepsilon(E_K) = \chi_K(-N)$.
(?)(CFT). where $\chi_K : \mathbb{Z}/|d_K| \longrightarrow \{\pm 1\}$ quad. char. associated to $K/\mathbb{Q}$.

Cor. If K satisfies (Heeg.) Then $\varepsilon(E_K) = -1$.

So (Heeg) $\Rightarrow \varepsilon(E_K) = -1 \Rightarrow r_{an}(E_K) = \text{odd}$.

$\Rightarrow y_K \in E(K)$?

Thm (Gross-Zagier)

$$L'(E_K, 1) = \frac{\int_{E(\mathbb{C})} \omega \wedge \overline{i\omega}}{|d_K|^{\frac{1}{2}} |\mathcal{O}_K^\times|^2} \langle y_K, y_K \rangle_{NT}.$$

Here $\omega \in H^0(E_{\mathbb{Q}}, \Omega^1)$ s.t. $\varphi^* \omega = 2\pi i f(z)\, dz$. normalized new form

Rmk:

$$\left(\int_{E(\mathbb{C})} \omega \wedge \overline{i\omega} \right) \cdot \deg \varphi = \int_{X_0(N)(\mathbb{C})} 8\pi^2 f(z) \overline{f(z)}\, \cancel{dz\, d\bar z}\, dx\, dy$$

$$= (f, f) \quad \text{(Peterson inner product)}.$$

So eqn:

$$L'(E_K, 1) = \frac{(f, f)}{|d_K|^{\frac{1}{2}} \left|\mathcal{O}_K^\times / \{\pm 1\}\right|^2} \cdot \frac{\langle y_K, y_K \rangle_{NT}}{\deg(\varphi)}$$

Rmk. y_K depends on choice of $\mathcal{N} \subseteq \mathcal{O}_K$.
(but determines y_K up to sign and ~~torsion~~).
and it also depends on $\varphi: X_0(N) \longrightarrow E$.
but $\dfrac{\langle y_K, y_K \rangle_{NT}}{\deg \varphi}$ is canonical!

Cor: $L'(E_K, 1) \neq 0 \iff \langle y_K, y_K \rangle_{NT} \neq 0$.
$\iff y_K$ is of infinite order.
so $r_{an}(E_K) = 1 \Rightarrow r_{alg}(E_K) \geq 1$.

Rmk comparing with BSD formula $\iff$

$$|Ш(E_K)|^{\frac{1}{2}} \overset{?}{=} \frac{[E(K) : \mathbb{Z} y_K]}{\prod_p c_p(E) \, |\mathcal{O}_K^\times|^2 \cdot c}$$

where c is the Manin constant (conj $=1$).
$\varphi^* \omega_E = c \cdot 2\pi i \, f_E(z) \, dz$.
In particular, if $\left. \begin{array}{l} p \nmid y_K \\ E(K)[p] = 0 \end{array} \right\} \overset{?}{\Rightarrow} Ш(E_K)[p^\infty] = 0$.

---

§2. Back to $E/\mathbb{Q}$.

Defn. Let $E^{(K)}/\mathbb{Q}$ be the quad. twist of E by K, i.e., unique elliptic curve $/\mathbb{Q}$ that is isom. to E over K but not isom. to E over $\mathbb{Q}$.
If $E: y^2 = x^3 + Ax + B$. Then $E^{(K)}: d_K y^2 = x^3 + Ax + B$.

Rmk: Consider $\rho_E : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(V_p E)$. Then $\rho_{E^{(K)}} \cong \rho_E \otimes \chi_K$.
where $\chi_K : G_\mathbb{Q} \twoheadrightarrow G_{K/\mathbb{Q}} \simeq \{\pm 1\}$.

Prop. $L(E_K, s) = L(E, s)\, L(E^{(K)}, s)$.

pf. check by defn.

$$L(E_K, s) = L\Big(\underbrace{\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \rho_E}_{\rho_E \oplus (\rho_E \otimes \chi_K)}, s\Big) = L(\rho_E, s) \cdot L(\rho_E \otimes \chi_K, s).$$ ∎

Cor. $r_{an}(E_K) = r_{an}(E) + r_{an}(E^{(K)})$.

Prop. $r_{alg}(E_K) = r_{alg}(E) + r_{alg}(E^{(K)})$.

pf. $$E(K) \otimes \mathbb{Q} \simeq E(\mathbb{Q}) \otimes \mathbb{Q} \oplus E^{(K)}(\mathbb{Q}) \otimes \mathbb{Q}$$

look at ± 1 of cplx conjugation.

Thm. If $r_{an}(E) = 1 \Rightarrow r_{alg}(E) \geq 1$.

pf. By a theorem of Waldspurger (next time), one can chose K satisfying (Heeg) s.t. $r_{an}(E^{(K)}) = 0$.

Then $r_{an}(E_K) = 1$.

$\Rightarrow r_{alg}(E_K) \geq 1$.

But $\varepsilon(E) = -1 \Rightarrow y_K^c = -\varepsilon(E)\, y_K = y_K$ in $\dfrac{E(K)}{E(K)_{tors}}$.

$\Longrightarrow P = y_K^c + y_K \in E(\mathbb{Q})$ is infinite order $\Rightarrow r_{alg}(E) \geq 1$.

Cor. If $r_{an}(E) = 1$, then $\dfrac{L'(E,1)}{\Omega(E) R(E)} \in \mathbb{Q}^\times$.

pf. It follows from $\int_{E(\mathbb{C})}$ $L'(E_K, 1) = L'(E,1) \cdot L(E^{(K)}, 1)$ — choose K as before.

Know: $\dfrac{L(E^{(K)}, 1)}{\Omega(E^{(K)})} \in \mathbb{Q}^\times$. Result follows from GZ formula.

using $\displaystyle\int_{E(\mathbb{C})} \omega \wedge \overline{i\omega} \sim_{\mathbb{Q}^\times} \Omega(E)\,\Omega(E^{(K)}) \cdot |d_K|^{\frac{1}{2}}$.

Cor. (of previous pf) If $\varepsilon(E) = -1$, and $y_K^c + y_K \in E(\mathbb{Q})$ is torsion. Then $r_{an}(E) \geq 3$.

using this can construct $E_{/\mathbb{Q}}$ with $r_{an}(E) = 3$.

Rmk. No example $E_{/\mathbb{Q}}$ w/ provably $r_{an}(E) = 4$.

Thm (Goldfeld) 1976. If there exists $E_{\mathbb{Q}}$ w/ $r_{an}(E) \geq 3$. Then

$$h(D) \gg C_{\delta,E}' \,(\log|D|)^{1-\delta}, \ \forall \delta > 0.$$

$h(D)$ = class # of $\mathbb{Q}(\sqrt{-D})$. $C_{\delta,E}'$ = effective const.

Ex. (Gauss elliptic curve) $y^2 + y = x^3 - 7x + 6$ $\quad N = 5077$ (is a prime # !)

~~Buhler - G - Z compute $y_K^c + y_K$ is torsion~~

$\Rightarrow r_{an}(E) \geq 3$.

Together w/ Goldfeld, this solves Gauss class # problem.

---

§3. Gross - Zagier formula for Shimura curves.

Fix $K/\mathbb{Q}$ imaginary quad. Define $N = N^+ \cdot N^-$ where $(N, d_K) = 1$

$p \mid N^+ \Longrightarrow p$ splits in K.

$p \mid N^- \Rightarrow p$ inert in K.

e.g. (Heeg) $\Rightarrow N = N^+$.

Assume N^- is square-free, then $\varepsilon(E_K) = -(-1)^{\#\{p \mid N^-\}}$.

so $\varepsilon(E_K) = \begin{cases} +1 & \text{if } \#\{p \mid N^-\} \text{ is odd} \\ -1 & \text{if } \#\{p \mid N^-\} \text{ is even.} \end{cases}$

Defn. K satisfies generalized Heegner hypothesis if N^- = sq-free product of <u>even</u> number of primes. (Heeg*).

$(\text{Heeg}^*) \Rightarrow \varepsilon(E_K) = -1 \overset{?}{\Longrightarrow} r_{an}(E_K) = \text{odd}$

$\overset{?}{\Longrightarrow}$ Heegner pts y_K $\overset{?}{\longleftrightarrow}$ $r_{an}(E_K)$

Since $\#\{p \mid N^-\}$ is even, we can construct

Defn. $B = B_{N^-}$ the unique quarternion alg. $/\mathbb{Q}$ ramified exactly over $\{p \mid N^-\}$.

Ex. $B = \mathbb{Q}\{1, i, j, ij\}$. $i^2 = a$, $j^2 = b$, $ij = -ji$.

if $a = 1$, then $B \simeq M_2(\mathbb{Q})$

$$i \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad j \longmapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \quad ij \longmapsto \begin{pmatrix} 0 & 1 \\ -b & 0 \end{pmatrix}$$

if $a = \text{prime } p \equiv 1 \ (4)$, $b = \text{prime } q$ s.t. $\left(\frac{q}{p}\right) \neq 1$ then $B = B_{\{p, q\}}$.

Defn. (analogue of $M_0(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid N \mid c\right\}$)

An Eichler order of level N^+ is a subring

$$\mathcal{O} \underset{\text{finite index}}{\subseteq} \mathcal{O}_B \leftarrow \text{max'l order in } B.$$

s.t. $$\mathcal{O}_p = \begin{cases} M_0(N^+)_p & p \nmid N^- \\ \mathcal{O}_{B_p} \leftarrow \text{max'l order} & p \mid N^- \end{cases}$$

Defn. (analogue of $\Gamma_0(N)$)

$\Gamma(N^+, N^-) = \{ \gamma \in \mathcal{O}^\times : \gamma\bar{\gamma} = 1 \}$.

(e.g. $N^- = 1$, $\Gamma(N^+, 1) = \Gamma_0(N^+)$).

Defn. Using $\Gamma(N^+, N^-) \hookrightarrow B^\times(\mathbb{R}) \simeq GL_2(\mathbb{R})$

$\xrightarrow{\gamma\bar{\gamma} = 1} SL_2(\mathbb{R}) \subseteq$.

We define the <u>Shimura curve</u> $X(N^+, N^-) := \Gamma(N^+, N^-) \backslash \mathcal{H}$.

Rmk
- If $N^- \neq 1$, $X(N^+, N^-)$ is already compact.
- This is an example of a Shimura variety.

One can rewrite it adelically:

$$K_p = \mathcal{O}_p^\times \subseteq B^\times(\mathbb{Q}_p) \text{ compact open subgp}$$

Take $K_f = \prod_p K_p \subseteq B^\times(\mathbb{A}_f)$. a compact open subgp.

Then $K_f \cap B_1^\times(\mathbb{Q}) = \Gamma(N^+, N^-)$. ($1$ ← norm 1.)

and

$$X(N^+, N^-) = \Gamma(N^+, N^-) \backslash \mathcal{H}$$
$$\cong B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f) \times \mathcal{H}^\pm / K_f.$$

Moduli Interpretation:
$$X(1, N^-)(\mathbb{C}) = \left\{ \begin{array}{c} A/\mathbb{C} \text{ abelian surface with} \\ \mathcal{O}_B \hookrightarrow \mathrm{End}(A) \end{array} \right\}.$$

$$\tau \in \mathcal{H} \longmapsto \mathbb{C}^2 / \mathcal{O}_B \cdot \binom{\tau}{1} \quad \text{via } \mathcal{O}_B \hookrightarrow M_2(\mathbb{C}).$$

$$X(N^+, N^-)(\mathbb{C}) = \left\{ \begin{array}{c} A/\mathbb{C} \text{ abelian surface w/} \\ A \longrightarrow A', \ \mathcal{O}_B\text{-isogeny + kernel is} \\ \text{cyclic } \mathcal{O}_B\text{-mod. of length } (N^+)^2 \end{array} \right\}$$

(e.g. $N^- = 1$, $A_\tau \simeq E_\tau \times E_\tau$).

Defn. Heegner points $x_K = (A \to A') \in X(N^+, N^-)(H_K)$
where $A \sim E^2$, $A' \sim E'^2$ and $\mathrm{End}(E) = \mathrm{End}(E') = \mathcal{O}_K$.

Defn. Using modular parametrization $\varphi : X(N^+, N^-) \longrightarrow E$
define $y_K = \sum_{\sigma \in \mathrm{Gal}(H_K/K)} \varphi(x_K^\sigma) \in E(K)$.

Thm (Yuan-Zhang-Zhang) Assume K satisfies (Heeg*). Then

$$L'(E_K, 1) = \frac{(f, f)}{|d_K|^{\frac{1}{2}} \left| \frac{\mathcal{O}_K^\times}{\{\pm 1\}} \right|} \cdot \frac{\langle y_K, y_K \rangle_{NT}}{\deg \varphi}.$$

---

§2. Waldspurger's formula. (case $\varepsilon(E_K) = +1$).

Assume (Waldspurger condition) N^- is square free product of an odd number of ~~pts~~ primes ($\Rightarrow \varepsilon(E_K) = +1$).

Goal: construct "y_K" s.t. $y_K \overset{?}{\longleftrightarrow} L(E_K, 1)$.

But $\#\{p \mid N^-\}$ is odd. Instead,

Defn. $B = B_{N^- \cup \{\infty\}}$ the unique quarternion alg. $/\mathbb{Q}$ ramified at $\{p \mid N^-\} \cup \{\infty\}$.

In particular: $B(\mathbb{R}) \simeq \mathbb{H} = \{ i^2 = -1, j^2 = -1, ij = -ji \}$.

$$B_1^\times(\mathbb{R}) \cong SU_2 = \left\{ \begin{pmatrix} x & -y \\ \bar{y} & \bar{x} \end{pmatrix} \in M_2(\mathbb{C}) : |x|^2 + |y|^2 = 1 \right\}.$$

is compact and does not act on $\mathcal{H}$.

Defn. Let $\mathcal{O} \subseteq \mathcal{O}_B$ be an Eichler order of level N^+. Define the Shimura Set

$$X(N^+, N^-) := B^\times(\mathbb{Q}) \backslash B^\times(\mathbb{A}_f) / K_f. \quad \text{where } K_f = \prod_p \mathcal{O}_p^\times$$

(0-dim'l Shimura vty).

Rmk. $B/\mathbb{Q}$ is called indefinite if $B(\mathbb{R}) \simeq M_2(\mathbb{R})$,

definite if $B(\mathbb{R}) \simeq \mathbb{H}$.

Recall $F = \#$ field. $F^\times \backslash \mathbb{A}_{F,f}^\times / \prod_p \mathcal{O}_{F,p}^\times \xrightarrow{\sim} Cl(\mathcal{O}_F)$.

Similarly: $X(1, N^-) \cong Cl(\mathcal{O}_B)$

$(= \{\text{right ideal classes of } \mathcal{O}_B\})$.

Defn. Fix an embedding $K \hookrightarrow B$ (s.t. $K \cap \mathcal{O} = \mathcal{O}_K$). Then we have

$$Cl(\mathcal{O}_K) \longrightarrow X(N^+, N^-) = Cl(\mathcal{O})$$
$$I \longmapsto I\mathcal{O}$$

Defn. A pt $x_K \in Cl(\mathcal{O}_K)$ is called a <u>CM point</u> (or Gross pt) on $X(N^+, N^-)$ ($Cl(\mathcal{O}_K)$ permutes these CM pts).

Defn. Let $f \in S_2^{new}(N)$ ~~whose~~ use Jacquet - Langlands :

get a function $\varphi : X(N^+, N^-) \longrightarrow \mathbb{C}$.

(they have the same Hecke eigenvalues).

Defn. $\cdot\ y_K = \sum_{x_K \in Cl(\mathcal{O}_K)} \varphi(x_K) \cdot |\mathrm{Aut}\, x_K| \in \mathbb{C}$. where $\mathrm{Aut}(x) = \{\gamma \in B^\times(\mathbb{Q}) : \gamma x = x\} / \{\pm 1\}$

$\cdot\ \deg \varphi = \sum_{x \in X(N^+, N^-)} |\varphi(x)|^2 \, (|\mathrm{Aut}(x)|)$.

Thm (Waldspurger) Assume K satisfies (Wald).

$$L(E_K, 1) = \frac{(f, f)}{|d_K|^{\frac{1}{2}} \left| \frac{\mathcal{O}_K^\times}{\{\pm 1\}} \right|^2} \cdot \frac{|y_K|^2}{\deg \varphi}.$$

Cor. $L(E_K, 1) \neq 0 \iff y_K \neq 0$.

y_K is called "Waldspurger's toric period".

Missed 2 Lectures!

§1. Inertia action on cohomology.

Motivation: - to understand $H^1_{sing}(K_\ell, E[p])$.

$$H^1_{sing}(K_\ell, E[p]) = H^1(I_\ell, E[p])^{Fr_\ell = 1}$$

$$I_\ell \circlearrowright E[p] = H^1_{ét}(E, \mathbb{F}_p)(1).$$

In general; switch notation

$X/\mathbb{Q}_p$ sm. proj vty; $K = \widehat{\mathbb{Q}_p^{unr}}$ (so: $I_p = Gal(\bar{K}/K)$).

Let $\mathcal{O} = \mathcal{O}_K \ (= \hat{\mathbb{Z}}_p^{ur})$ k = residue field $= \bar{\mathbb{F}}_p$.

$\Lambda = \mathbb{Z}/\ell$ or $\mathbb{Q}_\ell$.

Recall: Let $P \leq I$ be the wild inertia subgp, i.e., $P = Gal(\bar{K}/K^t)$ ← max'l tamely ramified extn of K

where $K^t = \bigcup_{p \nmid n} K(p^{\frac{1}{n}})$.

Then $I/p \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$.

Defn. Let $t : I \longrightarrow \mathbb{Z}_\ell(1)$ given by $\sigma \longmapsto \left\{ \dfrac{\sigma(p^{\frac{1}{\ell^n}})}{p^{1/\ell^n}} \right\}_n \in \mathbb{Z}_\ell(1)$

(the ℓ-adic tame character).

$\Lambda = \mathbb{Q}_\ell$.

Thm (Grothendieck). Let $V = H^i_{ét}(X_{\bar{K}}, \Lambda) \circlearrowleft I$. Then $\exists$ open subgp $J \subseteq I$ and a nilpotent matrix $N : V(1) \longrightarrow V$ s.t. $\forall \sigma \in J$, the action of $\sigma \circlearrowright V$ is given by

$$\exp(t(\sigma) N). \quad \text{(unipotent)}.$$

Defn. N is called the <u>monodromy</u> operator.

Example: Let $X/\mathbb{Q}_p$ be an elliptic curve w/ multiplicative reduction.
By Tate's uniformization, $\sigma = \begin{pmatrix} 1 & t(\sigma) \\ & 1 \end{pmatrix}$.
$\circlearrowright H^1_{et}(E_{\bar{K}}, \Lambda)$.

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

§2. Nearby cycle sheaves and monodromy filtration.

Goal: describe $H^i_{et}(X_{\bar{K}}, \Lambda)$ and N using the geometry of $X_k = X_{\bar{\mathbb{F}}_p}$.

Let $X_{\mathcal{O}}$ be ~~a proper integral model of~~ X_K.

$$X_k \xhookrightarrow[i]{} X_{\mathcal{O}} \xhookleftarrow[j]{} X_{\bar{K}}.$$

Defn. Nearby cycle is $R\Psi(\Lambda) := i^*Rj_*\Lambda \in D(X_k)$. ~~$\Lambda \to Rj_*\Lambda$~~

~~Then~~, $H^i(X_{\bar{K}}, \Lambda) = H^i(X_{\mathcal{O}}, Rj_*\Lambda)$
$= H^i(X_k, i^*Rj_*\Lambda)$.
$\simeq H^i(X_k, R\Psi\Lambda)$

Example: If $X_{\mathcal{O}} \to \mathrm{Spec}(\mathcal{O})$ is smooth, then $R\Psi\Lambda \simeq \Lambda$ and
$H^i(X_{\bar{K}}, \Lambda) \simeq H^i(X_k, \Lambda)$.
In this case, $N = 0$.

In general, Have an action $N: R\Psi\Lambda(1) \longrightarrow R\Psi\Lambda$.
which is nilpotent ($N^{n+1} = 0 \quad n = \dim X$).

Defn. ① kernel filtration: $F_i = \ker N^{i+1}$.

$$0 \subseteq F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = R\Psi\Lambda.$$

② image filtration $G^j = \operatorname{im} N^j$

$$R\Psi\Lambda \supseteq G^1 \supseteq G^2 \cdots \supseteq G^n \supseteq 0.$$

③ monodromy filtration $M_r := \sum_{i-j=r} F_i \cap G^j$.

$$0 \subseteq M_{-n} \subseteq \cdots \subseteq M_n = R\Psi\Lambda.$$

characterized by (1) $N(M_r(1)) \subseteq M_{r-2}$.

(2) $N^r : gr_r^M R\Psi\Lambda(r) \xrightarrow{\sim} gr_{-r}^M R\Psi\Lambda$.

§3. Rapoport-Zink (weight) spectral sequence

Defn. $X_{\mathcal{O}}$ is <u>semistable</u> if

(1) $X_{\mathcal{O}}$ is regular and flat$/\mathcal{O}$

(2) X_k is a divisor of $X_{\mathcal{O}}$ with normal crossings.

Further say $X_{\mathcal{O}}$ is <u>strictly semi-stable</u> if

(3) all irred. components of X_k are smooth.

Defn. Assume $X_{\mathcal{O}}$ is strictly semistable, then

$$X_k = X_1 \cup \dots \cup X_m$$

Define: for any subset $J \subseteq \{1, \cdots, m\}$ $X_J := \bigcap_{i \in J} X_i$.

For any $p \geq 0$ let $X^{(p)} := \coprod_{\substack{J \in \{1,\cdots,m\} \\ |J| = p+1}} X_J$.

Have restriction: $H^q(X^{(p)}, \Lambda) \longrightarrow H^q(X^{(p+1)}, \Lambda)$.

chern class map: $H^q(X^{(p)}, \Lambda)(-1) \longrightarrow H^{q+2}(X^{(p+1)}, \Lambda)$.

Thm. (Rapoport–Zink, Saito) Assume $X_{\mathcal{O}}$ strictly semistable. Then we have a spectral sequence:

$$E_1^{p,q} = H^{p+q}(X_k, \mathrm{gr}^M_{-p} R\Psi\Lambda) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda)$$

degenerates at E_2 if $\Lambda = \mathbb{Q}_\ell$.

Moreover $E_1^{p,q} = \bigoplus_{\substack{i\geq 0 \\ \& \, i \geq -p}} H^{q-2i}(X^{(p+2i)}, \Lambda)(-i)$.

where $d_1^{p,q}$ is a ~~sum~~ of res/cycle class maps,

and $N: E_1^{p+q} \longrightarrow E_1^{p+2,q-2}$ is given by $\otimes t$.

Conj. (wt-monodromy) $\mathrm{gr}^M_r H^i(X_{\bar{K}}, \Lambda)$ is pure of wt $i+r$.

Recall: $E_1^{p,q} = \bigoplus_{\substack{i\geq 0 \\ i\geq -p}} H^{q-2i}(X^{(p+2i)}, \Lambda)(-i) \Rightarrow H^{p+q}(X_{\bar{K}}, \Lambda)$.

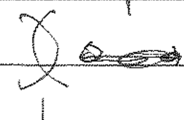
Notice $H^{q-2i}(X^{(p+2i)}) \neq 0$

$\Rightarrow$ ① $0 \leq p+2i \leq d := \dim X$

② $0 \leq q-2i \leq 2(d-(p+2i))$.

In particular: ① + $\begin{cases} p+2i \geq p \\ p+2i \geq -p \end{cases} \Rightarrow -d \leq p \leq d$.

② $\Rightarrow$ $\left.\begin{matrix} q+2i \leq 2d \\ q-2i \geq 0 \end{matrix}\right\} \Rightarrow q \in [0, 2d]$

Example Take X = elliptic curve w/ multiplicative reduction. Assume $\mathcal{X}_0$ has type I_2 reduction:  $\mathcal{X}_0 = \mathbb{P}^1 \cup \mathbb{P}^1$.

$\downarrow$

$\operatorname{Spec} \mathbb{F}_p$

$X^{(0)} = \mathbb{P}^1 \amalg \mathbb{P}^1$. $\quad X^{(1)} = pt \amalg pt$.

$p \in [-1, 1]$.

$$E_1: \quad \begin{tikzcd} \Lambda(-1)^{\oplus 2} \arrow[r, "\beta"] \arrow[ddrr, "N"] & \Lambda(-1)^{\oplus 2} & \\ & 0 & \\ & \Lambda^{\oplus 2} \arrow[r, "\alpha"] & \Lambda^{\oplus 2} \end{tikzcd}$$

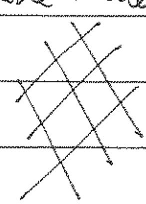
$$E_2: \quad \begin{tikzcd} \ker(\beta) \arrow[ddrr, "N"] & \operatorname{coker}(\beta) & 0 \\ 0 & 0 & 0 \\ 0 & \ker(\alpha) & \operatorname{coker}(\alpha) \end{tikzcd}$$

$$\rightsquigarrow 0 \subseteq \underset{\uparrow \atop \operatorname{coker}(\alpha)}{\mathrm{Fil}^0_1 H^1(X_{\bar K})} \subseteq \underset{\uparrow \atop \ker(\beta)}{H^1(X_{\bar K})} \rightsquigarrow$$

$$\begin{tikzcd} 0 \arrow[r] & \Lambda \arrow[r] & H^1(X_{\bar K}, \Lambda) \arrow[r] & \Lambda(-1) \arrow[r] \arrow[ll, bend left, "N"] & 0 \end{tikzcd}$$

Example Let X = Shimura curve $X(N^+, N^-)$, $p \| N^-$. (even # of primes)

Recall: it has a strictly semistable model with

$X_k =$  union of 2 families of $\mathbb{P}^1$'s.

$= X_1 \cup X_2$

$\mathbb{P}^1 \searrow \quad \swarrow \mathbb{P}^1$

$X' = X(N^+, \frac{N^-}{p})$. (Shimura set).

Both X_1, X_2 are disjoint union of $\mathbb{P}^1$'s indexed by X'.

$$X^{(0)} = X_1 \amalg X_2 \qquad X^{(1)} = \underset{\substack{\| \\ X(pN^+, \frac{N^-}{p})}}{X_1 \cap X_2} \xrightarrow{\deg(p+1)} X^{\cdot}$$

$E_1^{p,q}$:

$H^0(X^{(1)})(-1) \xrightarrow{\beta}$	$H^2(X_1 \amalg X_2)$	0
0	0	0
0	$H^0(X_1 \amalg X_2) \xrightarrow{\alpha}$	$H^0(X^{(1)})$

$E_2^{p,q}$:

$\ker\beta$	$\operatorname{coker}\beta$	0
0	$\cong$ N	0
0	$\ker\alpha$	$\operatorname{coker}\alpha$

<u>Want</u>: compute $H^1_{sing}(\underset{\substack{\| \\ \mathbb{Q}_p}}{K}, H^1(X_{\bar K})) = H^1(I, H^1(X_{\bar K}))^{Frob=1}$.

$$= \left(\frac{H^1(X_{\bar K})}{N(H^1(X_{\bar K}))}\right)^{Frob=1}.$$

$$\cong \frac{\operatorname{coker}(\alpha)}{N(\ker(\beta))}.$$

$$H^0(X_1 \amalg X_2) \xrightarrow{\alpha} H^0(X^{(1)}) \xrightarrow{\beta(1)} H^2(X_1 \amalg X_2)(1)$$

$\begin{pmatrix} p+1 & T_p \\ T_p & p+1 \end{pmatrix}$ indexed by $|X^{\cdot}|$.

$$= \frac{H^0(X^{(1)})}{Im(\alpha) + N(\underset{\substack{\| \\ \ker(\beta(1))}}{\ker\beta})}.$$

(*). Now localize at a max'l ideal $m \subseteq T$ non-Eisenstein s.t. $T_p \equiv \pm(p+1) \pmod{\ell}$.

(?). Also, after localization β is surj. & α is inj.

$$\Rightarrow H^1_{sing}(\mathbb{Q}_p, H^1(X_{\bar K})) = \frac{H^2(X_1 \amalg X_2)(1)}{im(\beta(1)\circ\alpha)} = H^2(X_1)(1). \text{ (as } \mathbb{T}\text{-mod.)}$$

Conclusion: $\underset{\text{Galois coh.}}{H^1_{sing}(\mathbb{Q}_p, H^1(X_{\bar K}))(1)_m} \cong \underset{\text{functions on Shimura set.}}{H^0(X^{\cdot})_m}$.

§1. First explicit reciprocity law.

Let $E/\mathbb{Q}$ elliptic curve. $K/\mathbb{Q}$ imaginary quadratic field.

Today: $r_{an}(E_K)=0$. Assume K satisfies (Wald): $N = N^+ \cdot N^-$

$p \mid N^+ \Rightarrow p$ splits in K, $p \mid N^- \Rightarrow p$ inert in K.

and (Wald): N^- is a square free product of <u>odd</u> number of primes.

Defn. Say a prime p is <u>nice</u> for (E, K) if

(1) $p \geq 5$: $p \nmid N \cdot d_K$.

(2) $\overline{\rho_{E,p}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[p]) = GL_2(\mathbb{F}_p)$ is surj. (Serre).

(3) $E(K_v)[p] = 0 \quad \forall\, v \mid N$ place of K.

(4) p is isolated: $H^0(X(\underset{\substack{\uparrow \\ \text{Shimura}}}{N^+}, N^-), \mathbb{Z}/p)_{\mathfrak{m}} \simeq \mathbb{Z}/p$.

Rmk: If E has no CM, then almost all p are nice.

Defn: Fix (E, K, p). Say ℓ is <u>admissible</u> if:

① $\ell \nmid pN$.

② ℓ is inert in K

③ ℓ is a level raising prime: $a_\ell \equiv \pm(\ell+1) \pmod p$.

④ $\ell \not\equiv \pm 1 \pmod p$.

Rmk: A positive proportion of ℓ are admissible.

Construction: Pick ℓ admissible: $E \rightsquigarrow f \in S_2(N) \overset{\text{Ribet}}{\equiv} g \in S_2(N\ell) \pmod p$

g ↑ level raised at ℓ.

$g \leadsto E(\ell)$ (in general is an abel. vty, assume elliptic curve for simplicity).

So: $E[p] \simeq E(\ell)[p]$

as $G_{\mathbb{Q}}$-rep.

$\varepsilon(E_K) = +1, \quad \varepsilon(E(\ell)_K) = -1.$

(1) Since K satisfies (Wald) for $X(N^+, N^-)$. We obtain a toric period y_K. Recall $y_K = \sum_{x_K \in Cl(\mathcal{O})} \varphi(x_K)$. where

$\varphi: \underset{X(N^+,N^-)}{\overset{\shortparallel}{Cl(\mathcal{O})}} \longrightarrow \mathbb{C}$ is the Jacquet-Langlands transfer of f.

Normalize φ by scaling such that φ has value in $\mathbb{Z}_p$ and $\varphi \bmod p$ is nonzero, then $y_K \in \mathbb{Z}_p$.

(2) Since ℓ is inert in K, K satisfies (Heeg*) for $X(N^+, \ell N^-)$.

$\rightsquigarrow$ Heegner pt $y_K(\ell) \in E(\ell)(K)$.

$\rightsquigarrow$ Kolyvagin class $c(\ell) \in H^1(K, E(\ell)[p]) = H^1(K, E[p])$.

have localization:

$$\begin{array}{ccc} H^1(K, E[p]) & \xrightarrow{loc_\ell} & H^1(K_\ell, E[p]) \\ & \searrow^{\partial_\ell} & \downarrow \\ & & H^1_{sing}(K_\ell, E[p]) \end{array}$$

Since ℓ is chosen s.t. $\ell \not\equiv \pm 1 \pmod p \Rightarrow E(K_\ell)[p] = \mathbb{Z}/p$.

and so $H^1_{sing}(K_\ell, E[p]) \cong \mathbb{Z}/p$.

$\rightsquigarrow \partial_\ell c(\ell) \in H^1_{sing}(K_\ell, E[p]) = \mathbb{Z}/p$.

so we obtain $\partial_\ell c(\ell) \in \mathbb{Z}/p$.

Thm. (Bertolini-Darmon) Assume p is nice, ℓ is admissible. Then $y_K \equiv \partial_\ell c(\ell) \pmod p$.

Rmk: $y_K \overset{\text{Waldspurger formula}}{\longleftrightarrow} L(E_K, 1) \qquad \partial_\ell c(\ell) \overset{\text{Kolyvagin}}{\longleftrightarrow} \text{~~Kolyvagin~~ } Sel_p(E_K)$

explicit reciprocity

pf. By computation of wt. s.s.:

$$H^1_{sing}(K_\ell, H^1(X(N^+, \ell N^-), \Lambda)_m(1)) \xrightarrow{\sim} H^2(X_1)_m(1)$$

$$X(N^+, \ell N^-) = X_1 \cup X_2 \qquad H^2(X_1)_m(1) \cong H^0(X_1)_m^* = H^0(X(N^+, N^-), \Lambda)_m^*$$

(parametrized by $X(N^+, N^-)$).

Look at: $\mathrm{Jac}(X(N^+, \ell N^-))(K) \longrightarrow H^1(K, E(\ell)[p])$

$$x_{K(\ell)} \longmapsto c(\ell), \qquad \partial_\ell : H^1(K, E(\ell)[p]) \to H^1_{sing}(K_\ell, E[p])$$

Since p is isolated: $H^0(X(N^+, N^-), \mathbb{Z}/p)_m = \mathbb{Z}/p \cong H^1_{sing}(K_\ell, E[p])$

$$x_K \longmapsto y_K = \sum \psi(x_K) \in \mathbb{Z}/p = H^0(X(N^+, N^-), \mathbb{Z}/p)_m^*$$

$$\mathrm{Div}(X_1) \longrightarrow \mathrm{Div}(X(N^+, N^-)) \longrightarrow \mathbb{Z}/p, \qquad \partial_\ell(c_\ell)$$

Fact: General formalism on $R\Psi\Lambda \Rightarrow$ this diagram commutes.

Therefore $y_K \equiv \partial_\ell c(\ell) \pmod p$.

§2. BSD in rk 0.

Thm. Assume p is nice.
If $y_K \not\equiv 0 \pmod p \Rightarrow \mathrm{Sel}_p(E_K) = 0$.

Cor. If $r_{an}(E_K) = 0 \Longrightarrow r_{alg}(E_K) = 0$ and $Ш(E_K)[p^\infty] = 0$ for almost all p.

pf. apply the thm to all p nice s.t. $y_K \not\equiv 0 \pmod p$. □

Rmk. One can also treat other missing primes by doing congruences mod p^m, $m \gg 0$.

Cor. If ~~E an~~ $r_{an}(E) = 0$, then $r_{alg}(E) = 0$ and $Ш(E)[p^\infty] = 0$ for almost all p.

pf. Pick K s.t. $r_{an}(E_K)=0$.

Sketch of pf of Thm. (application of Kolyvagin's method).

Assume $0 \neq s \in Sel_p(E_K)$. Then $\exists$ admissible prime ℓ, s.t. $loc_\ell(s) \neq 0$. (Chebotarov density). [BD, Section 3].

We obtain a class $c(\ell) \in H^1(K, E[p])$ s.t.

$$\partial_\ell c(\ell) \underset{\substack{\uparrow \\ \text{explicit reciprocity law}}}{\equiv} \mathcal{J}_K \not\equiv 0 \pmod{p}.$$

We have local conditions:

① $v \nmid pN\ell$ the $loc_v(s)$, $loc(c(\ell)) \in H^1_f(K_v, E[p])$.

② $v \mid N$ $H^1(K_v, E[p]) = 0$ ($E(\mathbb{Q}_v)[p]=0$ by p nice).

③ $v \mid p$ $loc_v(s)$, $loc_v(c(\ell)) \in H^1_f(\mathcal{O}_v, \mathcal{E}[p])$.

④ $v=\ell$, $loc_v(s) \in H^1_f(K_\ell, E[p]) = \mathbb{Z}/p$.

$\cancel{0 \neq \partial_\ell(c(\ell)) \in H^1_{sing}(K_\ell, E[p]) = \mathbb{Z}/p}$.

Then $0 \overset{CFT}{=} \langle s, c(\ell) \rangle = \sum_v \langle loc_v(s), loc_v\, c(\ell) \rangle$

$= \langle loc_v(s), loc_\ell(c(\ell)) \rangle \neq 0$.

— X —.

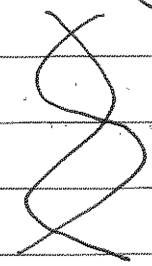
So $Sel_p(E_K) = 0$! □.

§1. Another example of weight spectral sequence.

Ex. Take $X = X(\ell N^+, N^-)$ Shimura curve. (N^- = even number of primes).
$\ell \nmid N^+$. (e.g. $N^-=1$)

It has semistable model $/\mathbb{Z}_\ell$.

$X_{\mathbb{F}_\ell} = X_1 \cup X_2$.



$X_1 \cong X_2 = X(N^+, N^-)$. $X^{(0)} = X_1 \amalg X_2$ $X^{(1)} = X_1 \cap X_2$ = supersingular locus.

$X^{(1)} = X(N^+, N^-)^{s.s.} = X(N^+, \ell N^-)$ Shimura set.

WSS:

$$\begin{array}{ccc} H^0(X^{(1)})(-1) \xrightarrow{\beta} & H^2(X_1 \amalg X_2) & 0 \\ 0 & H^1(X_1 \amalg X_2) & 0 \\ 0 & H^0(X_1 \amalg X_2) \xrightarrow{\alpha} & H^0(X^{(1)}) \end{array}$$

Localize at non-Eisenstein ideal m, then:

$$H^2(X_1 \amalg X_2)_m = 0. \quad H^0(X_1 \amalg X_2)_m = 0.$$

So $\cancel{\mathrm{coker}(\alpha)_m = H^0(X^{(1)})_m}$

$$\ker(\beta)_m = H^0(X^{(1)})_m(-1).$$

at E_2 (localized):

$$\begin{array}{ccc} H^0(X^{(1)})_m(-1) & 0 & 0 \\ 0 & H^1(X^{(0)})_m & 0 \\ 0 & 0 & H^0(X^{(1)})_m. \end{array}$$

get 2 steps filtration w/ successive quotients
(on $H^1(X(\ell N^+, N^-))_m$).

$$0 \subseteq \mathrm{Fil}_1^{\bullet} \subseteq \mathrm{Fil}_2 \subseteq \mathrm{Fil}_3 = H^1(X(\ell N^+, N^-))_m.$$

$$H^0(X^{(1)})_m(-1) \qquad H^0(X^{(1)})_m$$

monodromy: $\mathrm{gr}_2 \xrightarrow{N} \mathrm{gr}_0$

So we have an isom. of $\mathbb{T}$-modules

$$\underbrace{H^1(X(\ell N^+, N^-)_m)}_{\substack{\text{modular forms of level} \\ \ell \cdot N^+ N^- = \ell N \\ S_2(\ell N),\ N^-\text{-new.}}} \cong \underbrace{H^0(X^{(1)})_m^{\oplus 2}}_{\substack{X(N^+, \ell N^-) \\ S_2(\ell N),\ \ell N^-\text{-new.}}} \oplus \underbrace{H^1(X_1)_m^{\oplus 2}}_{\substack{X(N^+, N^-) \\ S_2(N),\ N^-\text{-new}}}.$$

Reformulation of Ihara's lemma: $\Lambda = \mathbb{Z}/p$.

Recall: (Ihara's Lemma) $H^1(X(N^+, N^-))_m^{\oplus 2} \xrightarrow{\pi_1^* + \pi_2^*} H^1(X(\ell N^+, N^-))_m$ is inj.

Or equivalently, by Poincare duality

$$H^1(X(\ell N^+, N^-))_m \xrightarrow{(\pi_{1,*},\, \pi_{2,*})} H^1(X(N^+, N^-))_m^{\oplus 2}.$$

Moreover: $(\pi_{1*}, \pi_{2*})(\mathrm{Fil}_1) = 0$. by Galois action considerati[on]

So we have a comm. diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^1(X_1 \amalg X_2) \arrow[r] \arrow[dr, bend right, "\begin{pmatrix} 1 & \mathrm{Fr}_\ell \\ \mathrm{Fr}_\ell & 1 \end{pmatrix}"'] & \dfrac{H^1(X(\ell N^+, N^-))}{\mathrm{Fil}_1} \arrow[r] \arrow[d, "(\pi_{1,*},\, \pi_{2,*})"', "\text{Ihara's lemma}"] & H^0(X^{(1)})(-1) \arrow[r] \arrow[d, "\gamma"] & 0 \\
& & H^1(X_1 \amalg X_2) \arrow[r] & \mathrm{coker}\begin{pmatrix} 1 & \mathrm{Fr}_\ell \\ \mathrm{Fr}_\ell & 1 \end{pmatrix} \arrow[r] & 0
\end{tikzcd}$$

So Ihara's lemma $\iff \gamma$ is surjective.

Now compute $\operatorname{coker}\begin{pmatrix} 1 & Fr_\ell \\ Fr_\ell & 1 \end{pmatrix} \cong \dfrac{H^1(X_1)}{1-Fr_\ell^2}$

$\cong H^1(\mathbb{F}_{\ell^2}, H^1(X_1)) = H^1_f(\mathbb{Q}_{\ell^2}, H^1(X(N^+,N^-)_{\bar{\mathbb{Q}}_\ell}))$

Cor. So Ihara's lemma $\iff H^0(X^{(1)})(-1) \xrightarrow{\gamma} H^1_f(\mathbb{Q}_{\ell^2}, H^1(X(N^+,N^-)_{\bar{\mathbb{Q}}_\ell}))$

Summary: $N^- =$ even product. Then

$H^0(X(N^+,\ell N^-))_{\mathfrak{m}} \longrightarrow H^1_f(\mathbb{Q}_{\ell^2}, H^1(X(N^+,N^-)_{\bar{\mathbb{Q}}_\ell})(1)_{\mathfrak{m}})$.

Last time: $N^- =$ odd product. Then,

$H^0(X(N^+,N^-))_{\mathfrak{m}} \xrightarrow{\sim} H^1_{sing}(\mathbb{Q}_{\ell^2}, H^1(X(N^+,\ell N^-)_{\bar{\mathbb{Q}}_\ell})(1)_{\mathfrak{m}})$.

§2. Second $\underbrace{\ }_{\text{explicit}}$ reciprocity law.

Let $E/\mathbb{Q}$ elliptic curve. Let K satisfies (Heeg*).

(i.e. $N^- =$ even product).

Assume p nice, ℓ admissible, then $\exists$ elliptic curve

$E(\ell)/\mathbb{Q}$ s.t. $E[p] \xrightarrow{\sim} E(\ell)[p]$.

$\varepsilon(E_K) = -1$ $\quad$ ~~$\varepsilon(E_K) = +1$~~ $\varepsilon(E(\ell)_K) = +1$.

E

$\downarrow$

Heegner pt $y_K \in E(K)$

Kolyvagin class $c(1) \in H^1(K, E[p]) \rightsquigarrow \mathrm{loc}_\ell\, c(1) \in H^1_f(K_\ell, E[p]) \xrightarrow{\sim} \mathbb{Z}/p$.

$E(\ell) \rightsquigarrow$ toric period $y_K(\ell) \in \mathbb{Z}_p$.

Thm (Bertolini–Darmon) $\mathrm{loc}_\ell(c(1)) \equiv y_K(\ell) \pmod{p}$.

$$\begin{array}{ccc} \mathrm{loc}_\ell(c(1)) & & y_K(\ell) \\ \updownarrow \text{Gross–Zagier} & & \updownarrow \text{Waldspurger} \\ L'(E_K,1) & \overset{\text{mod } p}{\equiv} & L(E(\ell)_K, 1) \end{array}$$

pf.

$$\begin{tikzcd} \mathrm{Jac}(X(N^+,N^-))(K) \arrow[r] \arrow[d, "\text{mod } \ell"'] & H^1(K, E[p]) \arrow[d, "\mathrm{loc}_\ell"] \\ & H^1_f(K_\ell, E[p]) = \mathbb{Z}/p \\ \mathrm{Div}(X(N^+,N^-)^{ss}) \arrow[r] & H^0(X(N^+,\ell N^-))_m = \mathbb{Z}/p \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} x_K \arrow[r, mapsto] \arrow[d, mapsto] & c(1) \arrow[d, mapsto] \\ x_K(\ell) \arrow[r, mapsto] & y_K(\ell) \arrow[r, leftrightarrow] & \mathrm{loc}_\ell(c(1)) \in H^1_f \end{tikzcd}$$

§3. BSD in rk 1.

Thm. Assume p is nice. If $0 \neq c(1) \in \mathrm{Sel}_p(E_K)$. Then $\mathrm{Sel}_p(E_K) = \mathbb{Z}/p$.

Cor. If $r_{an}(E_K) = 1$. Then $r_{alg}(E_K) = 1$ and $Ш(E_K)[p^\infty] = 0$ for almost all p.

pf. $0 \neq c(1)$ for almost all nice p. $\square$

Cor. If $r_{an}(E) = 1$, then $r_{alg}(E) = 1$ and $Ш(E)[p^\infty] = 0$ for almost all p.

pf. Pick K satisfy (Heeg*) s.t. $r_{an}(E_K) = 1$ $\square$

pf of Thm: Let $0 \neq s \in \mathrm{Sel}_p(E_K)$. Want: s is a scalar $\cdot\, c(1)$.

Pick an admissible prime ℓ_1 s.t.

$$0 \neq \mathrm{loc}_{\ell_1}(\cancel{\quad} c(1)) \in (\mathbb{Z}/p)^\times$$

So by replacing s by $\lambda s - c(1)$, ~~__~~ we may assume

$\mathrm{loc}_{\ell_1}(s) = 0$. WTS: $s = 0$.

Pick another admissible prime ℓ_2, s.t.

$$0 \neq \mathrm{loc}_{\ell_2}(s).$$

On the other hand, $E[p] \xrightarrow{\sim} E(\ell_1)[p] \xrightarrow{\sim} E(\ell_1\ell_2)[p]$

$\varepsilon = -1$ $\qquad \varepsilon = +1$ $\qquad \varepsilon = -1$

$c(1)$ $\qquad y_K(\ell_1)$ $\qquad c(\ell_1\ell_2)$.

$$0 \neq \mathrm{loc}_{\ell_1}(c(1)) \underset{\substack{\uparrow \\ \text{second explicit} \\ \text{reciprocity}}}{\equiv} y_K(\ell_1) \underset{\substack{\uparrow \\ \text{first explicit} \\ \text{reciprocity}}}{\equiv} \partial_{\ell_2}(c(\ell_1\ell_2)) \Rightarrow \partial_{\ell_2}(c(\ell_1\ell_2)) \neq 0.$$

$H^1_f \qquad H^1_{sing}$

$$\text{Finally compute } \underset{\substack{\uparrow \\ \text{CFT}}}{0 =} \langle s, c(\ell_1\ell_2)\rangle = \langle \mathrm{loc}_{\ell_2}(s), \partial_{\ell_2} c(\ell_1\ell_2)\rangle \neq 0.$$

Contradiction! So $s = 0$ & $\mathrm{Sel}_p(E_K) = \mathbb{Z}/p$.

Rankin–Selberg L-fctns for $GL_n \times GL_{n-1}$

Motivation: $L(E, s) = L(f, s)$

$$\begin{array}{ccc} L(E,s) & = & L(f,s) \\ \| & & \| \\ L(\rho_E, s) & & L(\pi, s) \end{array}$$

π automorphic repn on $GL_2(\mathbb{A})$ associated w/ f $(f \in \pi)$.

"motivic L-fctn" $\xrightarrow{\text{Langlands}}$ "automorphic L-fctn"

Goal: Study generalization to arithmetic of $L(s, \pi \times \pi')$ where π is a cuspidal automorphic repn on $GL_n(\mathbb{A})$, π' is a cuspidal automorphic repn on $GL_{n-1}(\mathbb{A})$.

Ex. $n=2$. $\left.\begin{array}{l} \pi = \pi \longleftrightarrow f \\ \pi' = \text{trivial} \end{array}\right\} \longleftrightarrow L(f, s)$.

Today: define $L(s, \pi \times \pi')$ and state its basic properties.

§1. Global zeta integrals

$\boxed{\mathbb{A} = \mathbb{A}_K, \ K/\mathbb{Q} \text{ \# field}}$

Defn. Let $f \in \pi$, $\varphi \in \pi'$, define

$$Z(f, \varphi, s) := \int_{[GL_{n-1}] = GL_{n-1}(K) \backslash GL_{n-1}(\mathbb{A})} f\begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi(g) |\det g|^{s-\frac{1}{2}} dg.$$

Example: $n=2$, $f \in S_2^{new}(N)$,

Recall: $L(f, s) = \sum_{n \geq 1} \frac{a_n}{n^s} = \prod_p L_p(f, s)$.

$$\Lambda(f, s) = (2\pi)^{-s} \cdot \Gamma(s) \cdot L(f, s)$$

$$= \int_0^\infty f(it)\, t^s \frac{dt}{t}.$$

$\mathcal{H} \ni x+iy \longleftrightarrow \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R})$.

$f \longmapsto f\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = f(iy) \cdot y$.

Then $\Lambda(f,s)$ generalizes to

$$\int_{[GL_1]} f\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} |y|^{s-1} dy.$$

$$\overset{\text{so}}{\Longrightarrow} \quad L(f,s) = Z(f, \mathbb{1}, s+\tfrac{1}{2}).$$

Thm (global properties of zeta integral)

① $Z(f, \varphi, s)$ converges to an analytic function on $s \in \mathbb{C}$

② $Z(f, \varphi, s)$ satisfies a functional equation

$$Z(f, \varphi, s) = Z(\tilde{f}, \tilde{\varphi}, 1-s)$$

where $\tilde{f}(g) = f({}^t g^{-1})$, similarly for $\tilde{\varphi}$.

③ Suppose $f = \otimes' f_v \in \pi = \otimes'_v \pi_v$

$\varphi = \otimes' \varphi_v \in \pi' = \otimes' \pi'_v$

Then $Z(f, \varphi, s) = \prod_v Z_v(f_v, \varphi_v, s)$.

to be defined.

pf. ①: $\because$ f, φ is a ~~are~~ cusp forms. (rapidly decreasing along cusps).

②: use the outer automorphism $g \longmapsto ({}^t g)^{-1}$.

③: Euler product uses uniqueness of ~~[illegible]~~ Whittaker models.

Whittaker expansion: Fix $\psi : K\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$

Defn. $f \in \pi$. define $W_f(g) = \int_{[N_n]} f(ng)\, \psi^{-1}(n)\, dn.$

$$N_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \in \cancel{SL_n}\, GL_n$$

where $\psi\begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ & & 1 \end{pmatrix} = \psi(x_{12} + x_{23} + \cdots + x_{n-1,n})$.

Thm
~~Jacquet~~
(Piateski-Shapiro,
Shalika)

$$f(g) = \sum_{\gamma \in \cancel{GL_{n-1}(K)} \, N_{n-1}(K)} W_f\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

Ex. $n=2$. $W_f(g) = \int_{[G_a]} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi^{-1}(x)\, dx$.

and $f(g) = \sum_{\gamma \in K^\times} W_f\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$.

$= \sum_{\gamma \in K^\times} W_{f,\psi_\gamma}(g)$ where $\psi_\gamma(x) = \psi(\gamma x)$.

($W_{f,\psi_\gamma}(g)$ is the generalization of $a_n \cdot q^n$).

Rmk: If $n>2$, difficulty for this thm arises b/c N_{n-1} is non-abelian. (do subtle induction on n).

Def: Let $\mathrm{Ind}_{N_n}^{GL_n}(\psi) = \left\{ W \in C^\infty(GL_n(\mathbb{A})) : \begin{array}{l} W(ng) = \psi(n) W(g) \\ n \in N_n(\mathbb{A}),\ g \in GL_n(\mathbb{A}) \end{array} \right\}$.

called the space of (global) Whittaker functions.

Then $W_f(g) \in \mathrm{Ind}_{N_n}^{GL_n}(\psi)$.

Defn. The (global) <u>Whittaker model</u>

$$W(\pi, \psi) := \{ W_f(g) : f \in \pi \} \circlearrowleft GL_n(\mathbb{A}).$$

We have a $GL_n(\mathbb{A})$ - equivariant map

$$\pi \longrightarrow W(\pi, \psi)$$
$$f \longmapsto W_f$$

Now using Whittaker functions to prove Euler product ("unfolding")

$$Z(f,\varphi,s) = \int_{[GL_{n-1}]} f\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \varphi(g) \, |\det(g)|^{s-\frac{1}{2}} dg.$$

$$= \int_{[GL_{n-1}]} \left(\sum_{\gamma \in N_{n-1}(K)\backslash GL_{n-1}(K)} W_f\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \cdot \varphi(g) \cdot |\det(g)|^{s-\frac{1}{2}} dg.$$

$$= \int_{N_{n-1}(K)\backslash GL_{n-1}(\mathbb{A})} W_f\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \varphi(g) \, |\det(g)|^{s-\frac{1}{2}} dg.$$

$$\xrightarrow[\text{property}]{\text{Whittaker fctn}} \int_{N_{n-1}(\mathbb{A})\backslash GL_{n-1}(\mathbb{A})} \underbrace{W_f\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_\varphi(g) \, |\det(g)|^{s-\frac{1}{2}}}_{\text{factorizable}} dg.$$

Defn. · $\mathrm{Ind}_{N_n}^{GL_n}(\psi_v) = \left\{ W \in C^\infty(GL_n(K_v)),\ W(ng) = \psi_v(n) \cdot W(g), \ n \in N_n(K_v),\ g \in GL_n(K_v) \right\}$

· A local Whittaker model is a nontrivial $GL_n(K_v)$ - equivariant map: $\pi_v \longrightarrow \mathrm{Ind}_{N_n}^{GL_n}(\psi_v)$.

(i.e. an elt in $\mathrm{Hom}_{GL_n(K_v)}(\pi_v, \mathrm{Ind}_{N_n}^{GL_n} \psi_v)$).

Thm. (uniqueness of local Whittaker model; by Gelfand-Kazhdan) Let π_v be a smooth admissible ~~irredu~~ irrep of $GL_n(K_v)$. Let $\psi_v : K_v \longrightarrow \mathbb{C}^\times$ character (nontrivial). Then,

$$\dim \mathrm{Hom}(\pi_v, \mathrm{Ind}_{N_n}^{GL_n}(\psi_v)) \leq 1.$$

(proof uses Bessel distributions).

<u>Thus</u>: If $f = \otimes f_v$, factorizable, then

$$W_f(g) = \prod_v \cancel{W} W_{f_v}(g_v).$$

Defn. $Z_v(f_v, \varphi_v, s) := \int_{N_{n-1}(K_v)\backslash GL_{n-1}(K_v)} W_{f_v}\begin{pmatrix} g_v & \\ & 1 \end{pmatrix} W_{\varphi_v}(g_v) |\det(g_v)|^{s-\frac{1}{2}} dg_v.$

(local zeta integrals)

Cor. $Z(f, \varphi, s) = \prod_v Z_v(f_v, \varphi_v, s).$

§2. Relation between $Z(f, \varphi, s)$ and $L(s, \pi \times \pi')$.

Thm (Local zeta integrals) ① $Z_v(f_v, \varphi_v, s)$ converges for $Re(s) >> 0$.

② $Z_v(f_v, \varphi_v, s) \in \mathbb{C}(q^{-s})$ and hence has meromorphic continuation to $\mathbb{C}$.

③. $\{Z_v(f_v, \varphi_v, s), f_v \in \pi_v, \varphi_v \in \pi_v'\}$ is ~~a fractional~~ an ideal over $\mathbb{C}[q^{\pm s}]$ in $\mathbb{C}(q^{-s})$. and hence has a generator $P_{\pi_v, \pi_v'}(q^{-s})^{-1}$, where $P \in \mathbb{C}[x]$.

normalize to ~~const~~. $P(0) = 1$.

Defn. $L(\pi_v \times \pi_v', s) := P_{\pi_v, \pi_v'}(q^{-s})^{-1}.$

Thm. $Z_v(f_v, \varphi_v, s) = \cancel{\omega_\pi} \, \omega_{\pi'}(-1)^{n-1} \gamma(s, \pi_v \times \pi_v', \psi_v) \cdot Z_v(\tilde{f}_v, \tilde{\varphi}_v, 1-s).$

(local fctional eqn)

Defn. $\varepsilon(s, \pi_v \times \pi_v', \psi_v) := \gamma(s, \pi_v \times \pi_v', \psi_v) \cdot \frac{L(s, \pi_v \times \pi_v')}{L(1-s, \tilde{\pi}_v \times \tilde{\pi}_v')}.$

§3. Global L-fctns.

Defn. $L(s, \pi\times\pi') = \prod_{v\nmid\infty} L(s, \pi_v \times \pi_v')$.

$\Lambda(s, \pi\times\pi') = \prod_v L(s, \pi_v\times\pi_v')$.

$\varepsilon(s, \pi\times\pi') = \prod_v \varepsilon(s, \pi_v\times\pi_v', \psi_v)$.

Thm. ① $L(s, \pi\times\pi')$ has analytic continuation to $\mathbb{C}$

② $\Lambda(s, \pi\times\pi') = \varepsilon(s, \pi\times\pi')\cdot\Lambda(-s, \tilde{\pi}\times\tilde{\pi}')$.

(use $Z(f, \varphi, s)$ and $Z_v(f_v, \varphi_v, s)$.

§1. General Waldspurger formula

Recall: $E/\mathbb{Q}$ elliptic curve, $K/\mathbb{Q}$ im. quad. satisfying

~~(Wald)~~ $\#\{p \mid N,\ p \text{ inert in } K\}$ is odd.

i.e. $\#\{p \mid N^-\}$ is odd.

(Wald) $\Rightarrow$ $\varepsilon(E_K) = +1$.

~~Waldspurger formula:~~

$$\frac{L(E_K, 1)}{(f,f)} = \frac{1}{|d_K|^{1/2}} \cdot \frac{|y_K|^2}{\deg \varphi}$$

toric period on definite quaternion alg.

$r_{an}(E)=0 \Rightarrow r_{alg}(E)=0$

$\Longrightarrow \left(L(E_K,1) \neq 0 \Longleftrightarrow y_K \neq 0\right)$. This is crucial in proving

~~Rem.~~ Notice: $L(E_K, s) \hat{=} L(\pi_K, s) = L(\pi_K \otimes \mathbb{1}_K, s)$.

$E \rightsquigarrow f \in S_2^{new}(N) \rightsquigarrow \pi$ aut. repn on $GL_2(\mathbb{A})$.
(with trivial central char.)

π_K = automorphic base change of π.

$\mathbb{1}_K : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ ~~trivial character~~.

Take F = # field. K/F quadratic extn.

B/F quaternion alg. (including $M_2(F)$).

π = cusp. aut. repn on $G := B^\times / F^\times$

χ = char. on $H := K^\times / F^\times$.

So an embedding $K \hookrightarrow B$ induces an embedding
$H \hookrightarrow G$ (as alg. gps / F).

Defn. The <u>automorphic H-period</u> is defined to be ($f \in \pi$).

$$\mathcal{P}_H(f) := \int_{[H] = H(F)\backslash H(\mathbb{A}_F)} f(h)\,\chi(h)\,dh \in \mathbb{C}.$$

(generalizing the toric period $\mathcal{P}_K$).

Notice: $\mathcal{P}_H : f \longmapsto \mathcal{P}_H(f)$ defines a linear functional

$$\mathcal{P}_H \in \mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \chi, \mathbb{C}) \quad \pi = \otimes' \pi_v, \ \chi = \otimes' \chi_v$$

$$= \prod_v{}' \mathrm{Hom}_{H(F_v)}(\pi_v \otimes \chi_v, \mathbb{C}).$$

Thm: (Tunnell, Saito)

1) (multiplicity one) $\dim \mathrm{Hom}_{H(F_v)}(\pi_v \otimes \chi_v, \mathbb{C}) \leq 1$.

2) (ε-dichotomy) Let $G^0(F_v) = PGL_2(F_v)$ π_v^0 irrep.

$G'(F_v) = B_v^\times / F_v^\times$. $\pi_v' = JL(\pi_v^0)$ ($\neq 0$ only when π_v^0 is a discrete series).

Then $\dim_{H(F_v)} \mathrm{Hom}(\pi_v^0 \otimes \chi_v, \mathbb{C})$

$+ \dim_{H(F_v)} \mathrm{Hom}(\pi_v' \otimes \chi_v, \mathbb{C}) = 1$.

Moreover, first term $\neq 0$ iff the local root number

$$\varepsilon(\tfrac{1}{2}, \pi_v^0 \otimes \chi_v) = \chi_v(-1) \cdot \eta_v(-1)$$

where $\eta = \eta_{K/F}$.

<u>assume</u>: $N(E)$ is square free.

Ex: π coming from $E/\mathbb{Q}$, $\chi = \mathbb{1}_K$. Then $\varepsilon(\frac{1}{2}, \pi_v \otimes \chi_v)$
$= \varepsilon_v(E/K)$

If $v \nmid N$, $\left.\begin{array}{l}\varepsilon_v(E/K) = 1 \\ \pi_v' = 0\end{array}\right\} \Rightarrow \dim \mathrm{Hom}_{H(F_v)}(\pi_v^0 \otimes \chi_v, \mathbb{C}) = 1.$

If $v \mid N$, then $\varepsilon_v(E/K) = \begin{cases} +1. & v \text{ splits in } K \\ (-1)^{\mathrm{ord}_v N}. & v \text{ inert in } K \end{cases}$

$\chi_v(-1) = 1$, $\eta_v(-1) = 1$ (since $v \nmid d_K$).

So $\dim_{H(F_v)}(\pi_v^0 \otimes \chi_v, \mathbb{C}) = 1$ when $v \mid N^+$

$\dim_{H(F_v)}(\pi_v^1 \otimes \chi_v, \mathbb{C}) = 1$ when $v \mid N^-$.

Hence (Walds) $\Rightarrow \dim \mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \chi, \mathbb{C}) = 1$.

(repn-theoretic interpretation of (Wald)).

Thm (Waldspurger) (non-vanishing criterion) TFAE:

1) $\mathcal{P}_H \neq 0$.

2) $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \chi, \mathbb{C}) \neq 0$. and $L(\pi \otimes \chi, \frac{1}{2}) \neq 0$.

Waldspurger also proves a formula for $L(\pi \otimes \chi, \frac{1}{2})$.

One construct a canonical elt $\alpha_v \in \mathrm{Hom}_{H(F_v)}(\pi_v \otimes \chi_v, \mathbb{C})$
$\otimes \mathrm{Hom}_{H(F_v)}(\tilde{\pi}_v \otimes \tilde{\chi}_v, \mathbb{C})$.
using matrix coefficients:

$f \in \pi_v$, $\tilde{f} \in \tilde{\pi}_v$ $\quad \alpha_v(f, \tilde{f}) := \displaystyle\int_{H(F_v)} \langle \pi_v(h) f, \tilde{f} \rangle \chi_v(h)\, dh$

Rmk: If π_v is unramified, $f, \tilde{f}$ spherical s.t. $\langle f, \tilde{f} \rangle = 1$,
then $\alpha_v(f, \tilde{f}) = \dfrac{\zeta_{F_v}(2) \cdot L(\frac{1}{2}, \pi_v \otimes \chi_v)}{L(1, \eta_v) \cdot L(1, \pi, Ad)} =: \mathcal{L}_v(\tfrac{1}{2}, \pi \otimes \chi)$.

Thm. (Waldspurger formula) Let $f = \otimes f_v \in \pi$, $\tilde{f} = \otimes \tilde{f}_v \in \tilde{\pi}$. Then,

$$\frac{\mathcal{P}_H(f) \cdot \mathcal{P}_H(\tilde{f})}{(f, \tilde{f})} = \frac{1}{4} \cdot \overbrace{\frac{\zeta_F(2)\, L(\frac{1}{2}, \pi \otimes \chi)}{L(1, \eta)\, L(1, \pi, Ad)}}^{\mathcal{L}(\frac{1}{2}, \pi \otimes \chi)} \cdot \prod_v \frac{\alpha_v(f_v, \tilde{f}_v)}{\mathcal{L}(\frac{1}{2}, \pi_v \otimes \chi_v) \cdot (f_v, \tilde{f}_v)}.$$

Rmk: $L(\frac{1}{2}, \pi \otimes \chi)$ may be thought of as the ratio of $\mathcal{P}_H(f)\, \mathcal{P}_H(\tilde{f})$ by "$\prod_v \alpha_v(f_v, \tilde{f}_v)$" in the 1-dim'l space!

§2. Gan-Gross-Prasad Conjectures

Waldspurger: $G = PGL_2 \cong SO_3 \quad H = K^\times/F^\times \cong SO_2$.

GGP generalizes this to many pairs of classical gps.

Our case: $G = U_n = U(V) \quad H = U_{n-1} = U(W)$.

where V is a Hermitian space of dim'n n wrt a quadratic extn K/F.

W = similar. of dim'n $n-1$ ... K/F.

Assume $V = W \oplus K\cdot u \quad (u,u)=1 \rightsquigarrow H \hookrightarrow G$

$h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$.

Goal: π = cusp. aut. tempered repn on G.

π' = ... H.

relate $\underbrace{L(\tfrac{1}{2}, \pi_K \times \pi_K')}_{\text{Rankin-Selberg } L\text{-fctn for } GL_n \times GL_{n-1}} \longleftrightarrow$ automorphic H-period $\mathcal{P}_H$.

write $\Pi = \pi \times \pi'$ $\longrightarrow$ aut. repn on $GL_n \times GL_{n-1}$.

Conj. (GGP)

① (multiplicity one) $\dim \mathrm{Hom}_{H(F_v)}(\Pi_v, \mathbb{C}) \leq 1$.

② (ε-dichotomy) $\sum_{(H^i, \pi_v')} \dim \mathrm{Hom}_{H^i(F_v)}(\Pi_v, \mathbb{C}) = 1$.

where (H^i, π_v') runs over Vogan L-packet of Π_v.

also has a description in terms of $\varepsilon(\Pi_v)$.

③ ~~(non-vanishing) TFAE:~~

1) $\mathcal{P}_H \neq 0$.

2) $\mathrm{Hom}_{H(\mathbb{A})}(\Pi, \mathbb{C}) \neq 0$ and $L(\tfrac{1}{2}, \Pi) \neq 0$.

④ (Ichino-Ikeda formula) $f \in \Pi$ $\tilde{f} \in \tilde{\Pi}$.

$$\frac{\mathcal{P}_H(f)\ \mathcal{P}_H(\tilde{f})}{(f, \tilde{f})} = \frac{1}{|S_{\Pi}|} \cdot \mathcal{L}\left(\tfrac{1}{2}, \Pi\right) \cdot \prod_v \frac{\alpha_v(f_v, \tilde{f}_v)}{\mathcal{L}(\frac{1}{2}, \Pi_v)(f_v, \tilde{f}_v)}$$

component gp of

~~L-packet~~

L-parameter of Π

Okay, these are known, due to very many people..........

§1. Bloch-Kato Selmer gps.

~~Motivation:~~ $E/\mathbb{Q}$ elliptic curve, we defined

$$\mathrm{Sel}_p(E) \subseteq H^1(\mathbb{Q}, E[p]). \quad \text{with}$$

$$0 \longrightarrow \frac{E(\mathbb{Q})}{pE(\mathbb{Q})} \longrightarrow \mathrm{Sel}_p(E) \longrightarrow Ш(E)[p] \longrightarrow 0.$$

More generally: $\mathrm{Sel}_{p^\infty}(E) := \operatorname{colim}_n \mathrm{Sel}_{p^n}(E) \cong \left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p}\right)^{r_p} \times$ finite

$$\mathrm{Sel}_{\mathbb{Q}_p}(E) := \lim_n \mathrm{Sel}_{p^n}(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbb{Q}_p^{r_p}.$$

Goal: define analogue of $\mathrm{Sel}_{\mathbb{Q}_p}(E)$ for more general Galois repn's.

$V/\mathbb{Q}_p$. finite dim'l vector space, $\rho: G_K \longrightarrow \mathrm{Aut}(V)$.

Bloch-Kato Selmer gp: $\boxed{H^1_f(K, V).}$???

$K = \#$ field.

Defn. A p-adic Galois repn $\rho: G_K \longrightarrow \mathrm{Aut}(V)$. <u>comes from Geometry</u> if it's a subquotient of $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p)(n)$ for i, n and some X/K smooth proj vty.

Rmk: Let K be a v-adic local field,

① if $v \nmid p$, X/K has good reduction, then.

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \cong H^i_{\acute{e}t}(X_{0,\bar{k}}, \mathbb{Q}_p) \text{ is } \underline{\underline{\text{unramified}}}.$$

② if $v \mid p$, then

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \text{ is } \underline{\underline{\text{de Rham}}}.$$

using $B_{dR}^{G_K} = K$, we know

$$\left(H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_K} \cong H^i_{dR}(X/K).$$

Def. $D_{dR}(V) := \left(V \otimes_{\mathbb{Q}_p} B_{dR}\right)^{G_K}$. V is <u>de Rham</u> if

$$\dim_K (D_{dR}(V)) = \dim_{\mathbb{Q}_p} V.$$

Defn: V is <u>geometric</u> if ① V is almost everywhere unramified.

② V is de Rham at all places $v \mid p$.

Hence V comes from geometry $\Rightarrow$ V is geometric.

Conj: (Fontaine-Mazur) V comes from geometry iff V is geometric.

Rmk: If $v \mid p$ and X/K has good reduction, then

$$H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p) \text{ is } \underline{\underline{\text{crystalline}}}.$$ (Beys, blablabla)

Thm (Faltings) $H^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys} \xrightarrow{\sim} H^i_{crys}(X_0/W(k)) \otimes_{W(k)} B_{crys}$.

$(B_{crys}^{G_K} = W(k)[\frac{1}{p}])$.

Defn. (local conditions) $H^1_f(K_v, V) \subseteq H^1(K_v, V)$.

① $v \nmid p$. define *unramified* subspace

$$H^1_f(K_v, V) := H^1(k_v, V^{I_v}) = \ker\left(H^1(K_v, V) \to H^1(I_v, V)^{F_v = 1}\right)$$

$\longrightarrow$ Then $\dim H^1_f(K_v, V) = \dim H^0(K_v, V)$.

(Gal(k_v) is pro-cyclic).

Also, local duality:

$$(\ ,\ )_v : H^1(K_v, V) \times H^1(K_v, V^*(1)) \longrightarrow H^2(K_v, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

$H^1_f(K_v, V)$ and $H^1_f(K_v, V^*(1))$ are exact annihilator

② $v \mid p$, define *finite* subspace:

$$H^1_f(K_v, V) := \ker\left(H^1(K_v, V) \to H^1(K_v, V \otimes_{\mathbb{Q}_p} B_{crys})\right).$$

Then: $\dim_{\mathbb{Q}_p} H^1_f(K_v, V) = \dim_{\mathbb{Q}_p} H^0(K_v, V) + \underbrace{\dim_{\mathbb{Q}_p} \frac{D_{dR}(V)}{D^+_{dR}(V)}}_{\#\text{ of negative Hodge-Tate wts.}}$.

Ex: $E/\mathbb{Q}$ elliptic curve, $V = V_p(E)$

Then ① $v \nmid p$, $H^1_f(K_v, V) = 0$.

② $v \mid p$, $\dim_{\mathbb{Q}_p} H^1_f(K_v, V) = 1$.

Also, local duality works.

Hence $H^1(K_v, V) = \begin{cases} 0 & \text{if } v \nmid p \\ 2 & \text{if } v \mid p. \end{cases}$ as V is self dual.

Defn. The <u>Bloch-Kato Selmer gp</u> is defined to be

$$H^1_f(K, V) := \{ s \in H^1(K, V) : \mathrm{loc}_v(s) \in H^1_f(K_v, V) \}$$

Thm (Bloch-Kato) E/K elliptic curve (or abelian variety), $V = V_p(E)$.
Then $H^1_f(K_v, V) = \mathrm{im}(A(K_v) \otimes \mathbb{Q}_p \longrightarrow H^1(K_v, V))$.
Thus $H^1_f(K, V) = \mathrm{Sel}_{\mathbb{Q}_p}(E)$.

---

§2. Bloch-Kato conj.

Defn. $L(s, V) := \prod_{v \nmid \infty} L_v(s, V)$ where

① $v \nmid p$, $L_v(s, V) := \det(1 - \mathrm{Fr}_v^{-1} q_v^{-s} \mid V^{I_v})^{-1}$.

② $v \mid p$, $L_v(s, V) := \det(1 - \varphi_v^{-1} q_v^{-s} \mid D_{\mathrm{crys}}(V))^{-1}$.

Example: $E/\mathbb{Q}$ elliptic curve, $V = V_p(E)$. Then

$$L(s, V) = L(s+1, E).$$

(center $= 0$).

Conj. (Langlands). Let V be a geometric Galois repn, then V should come from an automorphic repn, So $L(s, V)$ has meromorphic continuation to $s \in \mathbb{C}$.
(analytic if V doesn't ~~contain~~ have $\mathbb{Q}_p(1)$ as a subquotient).
and satisfies a functional eqn:

$$\Lambda(s, V) = \varepsilon(s, V) \cdot \Lambda(s, V^*(1)).$$

V irr. & $\neq \mathbb{Q}_p(1)$

0 if

Conj (Bloch-Kato). $\operatorname{ord}_{s=0} L(s, V) = \dim_{\mathbb{Q}_p} H^1_f(K, V^*(1)) - \dim_{\mathbb{Q}_p} H^0(K, V^*(1))$

(generalizes BSD).

---

§3. Rankin-Selberg motives.

Goal: ~~Defn.~~ • π on $GL_{n,K}$, K/F CM extn of a ~~totally~~ real field F.

π' on $GL_{n-1,K}$ $\quad (\Pi = \pi \times \pi')$.

• To construct a geometric Galois repn

$$V_{\Pi} = V_{\pi} \otimes V_{\pi'} \quad (\text{dim'n } n(n-1)).$$

Defn. Let π be a cusp auto. repn on $GL_n(\mathbb{A}_K)$.

Say (1) π is cohomological (= regular algebraic) if the infinitesimal character of π_∞, $\chi_\infty \in X^*(T)$ is reg, i.e., $\langle \chi_\infty, \alpha^\vee \rangle \neq 0 \quad \forall \alpha \in \Phi(GL_n)$.

(Hodge-Tate weights are distinct).

(2) conjugate self dual if $\pi^c \cong \tilde{\pi}$.

Next: construct V_π under these assumptions.

§1. BBK for Rankin-Selberg motives.

$K/F = \mathrm{CM}/\text{totally real}$. and $V^c \cong V^*(1)$.

$\hookleftarrow$ geometric p-adic Galois repn.

Then $L(s,V) = L(s,V^c) = L(s, V^*(1))$.

Hence fctional eqn: $L(s,V) \longleftrightarrow L(-s,V)$, center $s=0$.

Now π = cusp. ~~ess~~ tempered automorphic repn on $GL_n(\mathbb{A}_K)$.

" $\rightsquigarrow$ V_π ".

Thm. (Chenevier-Harris, Shin + many people). Assume π is cohomological and conjugate self-dual. ($\pi^c \simeq \tilde{\pi}$).

Then $\exists$ $\rho_\pi : G_K \to \mathrm{Aut}(V_\pi)$ s.t. ← p-adic Galois repn

① V_π is geometric, de Rham at $v|p$ w/ distinct HT wts.

② If π_v is unramified, then V_π is $\begin{cases} \text{unramified} & v \nmid p \\ \text{Crystalline} & v|p. \end{cases}$

$\pi_v \xleftrightarrow{\text{Satake}}$ Satake parameter $C_{\pi_v} = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \in GL_n(\mathbb{C})$.

The eigenvalues of $\underset{(v\nmid p)}{Fr_v}$ or $\underset{(v|p)}{\varphi_v}$ are given by $\{\mu_i q^{\frac{1-n}{2}}\}$, $|\mu_i| = 1$.

In other words $V_\pi|_{G_v} = LLC(\pi_v \otimes |\cdot|_v^{\frac{1-n}{2}})$.

③. More generally $V_\pi|_{G_v}$ is cptible w/ $LLC(\pi_v)$.

In particular ~~$V_\pi^c = V_\pi$~~ $V_\pi^c = V_\pi^*(1-n)$.

Idea of pf: When n = odd, or n = even + π_∞ satisfies extra regularity and V_π can be constructed in the coh. of certain unitary Shimura

$\mathbb{Q}$

variety. (Shin).

① pt counting on Igusa vties.

② compare Lefschetz trace formula & Arthur Selberg trace formula (eigenvarieties).

n = even is removed by Chenevier – Harris by congruences. ▨

Defn. (Rankin-Selberg motive) π on $GL_n(\mathbb{A}_K)$ as before, π' on $GL_{n-1}(\mathbb{A}_K)$ as before. (same assumption)

Define R-S motive, $V_{\Pi} := (V_{\pi} \otimes V_{\pi'})(n-1)$, $\Pi = \pi \times \pi'$.

Then $V_{\Pi}^c = V_{\Pi}^*(1)$. and $L_v(s, V_{\Pi}) = \det(1 - q_v^{-s} \mid C_{\pi_v} \otimes C_{\pi'_v})^{-1}$ (Satake)

$$= \prod_{i,j} (1 - q_v^{-s} \cdot \mu_i \cdot \mu_j')^{-1}.$$

Thm (LTXZZ) rk=0. Assume π, π' have trivial infinitesimal char. (Hodge-Tate $\pi: \{0, -1, -2, \cdots, -(n-1)\}$, $\pi': \{0, -1, -2, \cdots, -(n-2)\}$)

Let V_{Π} be the Rankin-Selberg motive.

Then $L(\frac{1}{2}, \Pi) \neq 0 \Rightarrow H^1_f(K, V_{\Pi}^*(1)) = 0$.

(i.e. BK conj holds in rk 0) (for almost all p).

§2. Rk 1 case.

<u>Want</u>: $\mathrm{ord}_{s=\frac{1}{2}}(L(s, \Pi)) = 1 \Rightarrow \dim H^1_f(K, V_{\Pi}^*(1)) = 1$.

In the elliptic curve case, this crucially relies on Gross-Zagier formula

$$L'(E_K, 1) \sim \text{height of Heegner pt } y_K.$$

<u>Generalization</u>: (Arithmetic Gan-Gross-Prasad conj.)

$$L'(\tfrac{1}{2}, \Pi) \sim \text{"height" of GGP cycle } \Delta_{\Pi}.$$

Goal: construct Δ_π (using Shimura variety associated w/ $U_n \times U_{n-1}$)

Defn: Let V = hermitian space wrt K/F of dim n. Let $U_n = U(V)$ associated unitary gp.

Assume $U_n(\mathbb{R}) = \overbrace{U(n-1;1) \times U(n;0) \times \cdots \times U(n;0)}^{[F:\mathbb{Q}]}$.

Defn: $h: \mathbb{C}^\times \longrightarrow U_n(\mathbb{R}) = U(n-1;1) \times U(n;0) \times \cdots \times U(n;0)$.

$$z \longmapsto \left(\begin{pmatrix} \bar{z}/z & & \\ & \ddots & \\ & & 1 \end{pmatrix}, Id, \ldots, Id \right).$$

$\mathcal{H} :=$ orbit of h under $U_n(\mathbb{R})$-conjugation.

$\cong U(n-1;1) / U(n-1) \times U(1)$.

It's a Hermitian symmetric domain, $\dim_{\mathbb{C}} \mathcal{H} = n-1$.

Defn: Let $K \subseteq U_n(\mathbb{A}_f)$ an open cpt subgp. Define Shimura vty associated w/ ..
$\mathbb{A}_f = \mathbb{A}_{F,f}$

$$Sh_K(U_n)(\mathbb{C}) = U_n(F) \backslash U_n(\mathbb{A}_f) \times \mathcal{H} / K.$$

Assume K has no torsion elts, then it's qproj vty & smooth $/F \subseteq \mathbb{C}$

$\dim = n-1$.

Rmk: Technically, it's easier to work w/ Shimura vty associated w/ $GU(V)$ instead of $U(V)$.

$$GU(V) = \{ g \in GL(V) : \langle gv_1, gv_2 \rangle = \underbrace{\chi(g)}_{\text{scalar} \in \mathbb{Q}^\times} \langle v_1, v_2 \rangle \}.$$

$$0 \longrightarrow U(V) \longrightarrow GU(V) \xrightarrow{\chi} \mathbb{G}_m \longrightarrow 0 \quad /\mathbb{Q}.$$

Consider $\mathbb{C}^\times \longrightarrow GU(\mathbb{R})^{(V)}$

$$z \longmapsto \left(\begin{pmatrix} \bar{z} & & & \\ & z & & \\ & & \ddots & \\ & & & z \end{pmatrix}, z\cdot Id, \cdots, z\cdot Id \right).$$

Then $Sh_K(GU(V))$ is of <u>PEL type</u>.

$$Sh_K(GU(V))(\mathbb{C}) = \{(A, \iota, \lambda, \eta)\}.$$

- A = abelian vty of dim $n-1$.
- $\iota : \mathcal{O}_K \hookrightarrow End(A)$
- $\lambda : A \xrightarrow{\sim} A^\vee$ $\mathcal{O}_K$-linear polarization.
- η : K-orbit of isomorphism $H^1(A,\mathbb{Q}) \otimes \mathbb{A}_f \overset{\eta}{\cong} V(\mathbb{A}_f)$ ~~$GU(V)(\mathbb{A}_f)$ of symplectic $\mathbb{A}_f$-mod $\otimes \mathcal{O}_K$~~ (up to scalar).

Rmk: $F \neq \mathbb{Q}$. $Sh_K(GU(V))$ is projective.
$F = \mathbb{Q}$ —— " — has a canonical toroidal cptification.
denote them by X_V. (cptification).

Defn. (GGP cycle) $V = W \oplus K\cdot u$ $(u,u)=1$.

$$GU(W) \hookrightarrow GU(V) \rightsquigarrow \underset{\dim n-2}{X_W} \overset{\delta}{\hookrightarrow} \underset{\dim n-1}{X_V}.$$

Consider: $(Id, \delta) : X_W \hookrightarrow X_W \times X_V$.
dim: $n-2$ $\quad 2n-3$.

GGP cycle $\Delta := im\,(Id \times \delta) \subseteq CH^{n-1}(X)$.

Given Π, define $\Delta_\Pi = \Delta[\Pi_f] \subseteq CH^{n-1}(X)$.

Rmk. local ε-dichotomy ensures there is a unique choice of (W,V) s.t. Δ_Π is possibly nonzero.

Conj. (Arithmetic GGP) $L'(\frac{1}{2}, \Pi) \sim$ "ht" of Δ_Π.

Rmk. AGGP is a special case of Beilinson's conj.:

X/K sm. proj vty.

Conj. (Beilinson): $\mathrm{ord}_{s=\text{center}} L(s, H^{2i-1}(X))$

$= \dim_{\mathbb{Q}} CH^i(X)_0 \left(= \cancel{CH^i} \ker\left(CH^i(X) \xrightarrow{cl} H^{2i}(X)_{(i)}\right)\right)$

Rmk: $L'(\frac{1}{2}, \Pi) \neq 0 \Rightarrow \Delta_\Pi \neq 0 \Rightarrow CH^{n-1}(X) \xrightarrow{cl} H^{2n-2}(X, \overline{\mathbb{Q}}_p)(n-1)[\Pi_f]$

$\Delta_\Pi \in$, $\quad H^{2n-2}(X_{\bar{K}}, \overline{\mathbb{Q}}_p)(n-1)[\Pi_f] = 0 \longrightarrow$

$\xrightarrow{\text{Abel-Jacobi}} H^1(K, H^{2n-3}(X_{\bar{K}}, \overline{\mathbb{Q}}_p)(n-1)[\Pi_f]) \ni AJ(\Delta_\Pi)$

Rmk: $n=2$, $AJ(\Delta_\Pi)$ is the analogue of Heegner pt y_K under Kummer map

$E(K) \to H^1(K, V_p(E))$.

Thm (LTXZZ) ($rk=1$). For almost all p:

$AJ(\Delta_\Pi) \neq 0 \Rightarrow \dim H^1_f(K, V_\Pi^*(1)) = 1$.

§1. Tate Conjecture via Geometric Satake.

Recall: new ingredient in proving explicit reciprocity (Tate conj. for unitary Shimura vty).

$X = X_{V, \bar{k}_v}$ special fiber associated w/ $GU(1, n-1)$.

Assume $\underline{n = \text{odd}} \Rightarrow \dim X = n-1$ is even.

Last time: (I missed it!)

$$X \longleftarrow \widetilde{X}^{ss} \quad \dim \tfrac{n-1}{2}$$

$\dim n-1$ $\quad \downarrow \longleftarrow$ Deligne-Lusztig vty of $\dim \frac{n-1}{2}$.

X° ~~= Shimura set.~~

Thm (Xiao-Zhu) Assume v is odd generic for π. Then,

$$H^{n-1}(\widetilde{X}^{ss})\left(\tfrac{n-1}{2}\right)[\pi_f] \simeq H^{n-1}(X)\left(\tfrac{n-1}{2}\right)^{Fr_v = 1}[\pi_f].$$

Rmk: This holds with coefficient integral: let $m_\pi \subseteq \mathbb{T}$ correspond to π, then

$$H^{n-1}(\widetilde{X}^{ss}, \Lambda)\left(\tfrac{n-1}{2}\right)_{m_\pi} \xrightarrow{\sim} H^{n-1}(X, \Lambda)\left(\tfrac{n-1}{2}\right)^{Fr_v = 1}_{m_\pi}.$$

(this follows from torsion-free result of Caraiani - Scholze).

pf strategy: ① injectivity: i.e. the image of components of $\widetilde{X}^{ss}$ are linearly independent in $H^{n-1}(X)$.

② compare dim (dim are same!): proved using comparison of Lefschetz trace formula for X° and X.

① is more difficult, look at the intersection pairing

$$H^{n-1}(X) \times H^{n-1}(X) \longrightarrow H^{2n-2}(X) \simeq \mathbb{Q}_p. \ (\mathbb{Q}_\ell ?)$$

and it suffices to show the matrix under $(\cdot, \cdot)$ for $H^{n-1}(\widetilde{X}^{ss})$ is non degenerate. But this is difficult to compute.

Langlands dual $\downarrow$ $\hat{G}$

Idea: compute the determinant abstractly using repn theory of $\hat{G}$.

- on the generic fiber: have an action of Hecke correspondence at v:

$$\mathcal{H}_v \xrightarrow[\sim]{\text{Satake}} \mathbb{C}[X_*(T)]^W \xrightarrow{\sim} \underset{(G \text{ split})}{\mathbb{C}[\hat{G}]^{\hat{G}}}$$

- on the special fiber: can realize

$$\begin{tikzcd} & \tilde{X}^{ss} \arrow[dl] \arrow[dr] & \\ X^{\circ} & & X \end{tikzcd}$$

as "exotic Hecke correspondence" between X° and X.
(has meaning in terms of "spectral operators").

§2. Geometric Satake.

$G/\mathcal{O}$ red. gp.

F = local non-arch. field, k = residue field, $\mathcal{O} = \mathcal{O}_F \subseteq F$.

Defn. Spherical Hecke alg. $\mathcal{H} := C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$.

Satake isomorphism: $\mathcal{H} \xrightarrow{\sim} \overline{\mathbb{Q}}_p[\hat{G}\sigma]^{\hat{G}}$ ← σ conjugation, σ = Frobenius
$g \cdot h = gh({}^{\sigma}g^{-1})$.

To geometrize this:

Defn: (local Hecke Stack) $Hk^{loc} := L^+G\backslash LG/L^+G$ where $LG(k) = G(F)$, $L^+G(k) = G(\mathcal{O})$.

For R a perfect k-alg, define $D_R = \mathrm{Spec}\, W_{\mathcal{O}}(R)$, $D_R^* = \mathrm{Spec}\, W_{\mathcal{O}}(R)[\frac{1}{\varpi}]$

$Hk^{loc}(R) = \{\mathcal{E}_1 \xrightarrow{\beta} \mathcal{E}_2 : \mathcal{E}_1$ and $\mathcal{E}_2$ are G-bundles on D_R and $\beta|_{D_R^*}$ is an isom$\}$.

I'm not sure...

Let ~~Perv~~ $\mathrm{Perv}(Hk^{loc})$ be the category of perverse sheaves on Hk^{loc}.

Thm (Mirković–Vilonen if char $F>0$, Xinwen Zhu if char $F=0$).

$\exists$ equivalence of monoidal cats:

$$\mathrm{Rep}(\hat{G}) \xrightarrow{\sim} \mathrm{Perv}(\mathrm{Hk}^{loc}).$$

$$\underset{\text{heighest wt } \mu}{V_\mu = \text{irrep of}} \longmapsto IC_\mu \quad (IC \text{ sheaf on } Gr_\mu \subseteq Gr).$$

§3. Spectral operators

Defn. (moduli of local shtukas)

$$\mathrm{Sht}^{loc} := \left\{ (\mathcal{E}_1 \xrightarrow{\beta} \mathcal{E}_2) \in \mathrm{Hk}^{loc} + \sigma^*\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2 \right\}.$$

by defn, we have $\mathrm{Sht}^{loc} \longrightarrow \mathrm{Hk}^{loc}$.

Defn.

$$\begin{tikzcd} & \mathrm{Hk}(\mathrm{Sht}^{loc}) \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Sht}^{loc} & & \mathrm{Sht}^{loc} \end{tikzcd} \qquad \mathrm{Hk}(\mathrm{Sht}^{loc}) := \left\{ \begin{tikzcd} \mathcal{E}_1 \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathcal{E}_2 \arrow[r, phantom, "\simeq"] \arrow[d, "\sigma^*\alpha"] & \sigma^*\mathcal{E}_1 \arrow[d] \\ \mathcal{E}_1' \arrow[r, "\beta'"] & \mathcal{E}_2' \arrow[r, phantom, "\simeq"] & \sigma^*\mathcal{E}_1' \end{tikzcd} \right\}$$

Let $\mathrm{Sht}^{loc}_\mu \subseteq \mathrm{Sht}^{loc}$ given by the locus where $\mathrm{inv}(\beta) \leq \mu$.

Then have correspondence:

$$\begin{tikzcd} & \mathrm{Sht}^{loc}_{\mu_1 | \mu_2} \subseteq \mathrm{Hk}(\mathrm{Sht}^{loc}) \arrow[dl] \arrow[dr] & \\ \mathrm{Sht}^{loc}_{\mu_1} & & \mathrm{Sht}^{loc}_{\mu_2} \end{tikzcd}$$

$\mathrm{inv}(\beta) \leq \mu_1$ and $\mathrm{inv}(\beta') \leq \mu_2$.

Example: $\mathrm{Sht}^{loc}_0 = [\mathrm{Spec}\, k / G(\mathcal{O})]$. $\quad \mathrm{Sht}^{loc}_{0|0} = [G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})]$

Defn: The category $\mathrm{Perv}^{Corr}(\mathrm{Sht}^{loc})$

1. objects: perverse sheaves on Sht^{loc}.
2. morphisms: cohomological correspondences supported on $\mathrm{Hk}(\mathrm{Sht}^{loc})$.

i.e. $\mathrm{Hom}^{Corr}(\mathcal{F}_1, \mathcal{F}_2) = \mathrm{Hom}(\overleftarrow{h}^* \mathcal{F}_1, \overrightarrow{h}^! \mathcal{F}_2)$.

Thm (Xiao–Zhu)

$\exists$ functor S s.t.

$$\begin{tikzcd} \mathrm{Coh}([\mathrm{pt}/\hat{G}]) = \mathrm{Rep}(\hat{G}) \arrow[r, "\text{Satake}", "\sim"'] \arrow[d, "\text{pullback}"] & \mathrm{Perv}(\mathrm{Hk}^{loc}) \arrow[d, "\text{pullback}"] \\ \mathrm{Coh}^{free}([\hat{G}\sigma/\hat{G}]) \arrow[r, "\exists S"] & \mathrm{Perv}^{Corr}(\mathrm{Sht}^{loc}) \end{tikzcd}$$

$$V \mapsto \tilde{V} = V \otimes \mathcal{O}$$

Rmk: Let V_0 be the trivial repn $\mathbb{1}$ of $\hat{G}$.

$$\mathrm{End}(\tilde{V}_0) = \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O}) = \bar{\mathbb{Q}}_p[\hat{G}\sigma]^{\hat{G}}$$

= twisted conj invariant fctns on $\hat{G}$.

$\left(\underset{\text{Satake}}{\simeq} \mathcal{H}\right)$.

More generally, define $J(V) = \Gamma([\hat{G}\sigma/\hat{G}], \tilde{V})$

$$= (\bar{\mathbb{Q}}_p[\hat{G}\sigma] \otimes V)^{\hat{G}}$$

$$= \{ f : \hat{G} \to V,\ f(gh\sigma(g)^{-1}) = g.f(h) \}.$$

Let $J := J(V_0)$ $(\simeq \mathcal{H})$.

§4. Computing intersection #.

Notice: there is a map

$$\underset{\substack{\| \\ Sh(G,\mu)}}{X} \longrightarrow Sht_\mu^{loc} \quad \text{and} \quad \begin{array}{ccccc} & & \tilde{X}^{ss} & & \\ & \swarrow & \downarrow & \searrow & \\ X^\circ & & Sht^{loc}_{0|\mu} & & X \\ \downarrow & \swarrow & & \searrow & \downarrow \\ Sht_0^{loc} & & & & Sht_\mu^{loc} \end{array}$$

and $\tilde{X}^{ss}$ can be identified via

$$H^0(X^\circ) \underset{J}{\otimes} \underbrace{\overset{J(V_\mu)}{\overset{\|}{\mathrm{Hom}(\tilde{V}_0, \tilde{V}_\mu)}}}_{\text{"exotic correspondence"}} \longrightarrow H^{n-1}(X).$$

via S-operator.

and the intersection matrix is given by

$$J(V_\mu) \times J(V_\mu^*) \longrightarrow J.$$

Upshot: (back to $G = U(n)$, n is odd)

$$\mu(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} \quad V_\mu = \text{Std of } GL_n.$$

Thm: 1) $J(V_\mu)$ is a free module of rk 1 over J.

2) The intersection matrix (1×1) is given by

$$\prod_{i=1}^{\frac{n-1}{2}} \left(X_i + X_i^{-1} - 2\right) \quad \text{where} \quad X_i(t) = \begin{array}{c} \\ \\ \end{array}\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & t & & \\ & & & \ddots & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & t^{-1} \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{pmatrix}$$

i-th

After localizing at $\mathfrak{m}_\pi$, get $\prod_{i=1}^{\frac{n-1}{2}} (\mu_i + \mu_i^{-1} - 2) \neq 0$

~~where $\{\mu_i\}$ Satake parameter of π~~ b/c $\mu_i \neq 1$.